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Curve Modeling with Jacobi Elliptic Functions

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Abstract. In this study, at first we analyze the construction method of basis functions for Multiquadratic Curves: MQ-Curves, which is an Extended Complete Tchebycheff System. The we apply the method for Jacobi elliptic functions as blending functions for free-form curve formulations. It is known that the shape of a rotating rope can be expressed by elliptic functions, and it is meaningful to use elliptic functions as blending functions. In this paper we focus on the usage of Jacobi elliptic functions, but our method proposed in this paper is general and we can adopt other types of special functions as blending functions.

Keywords: curve modeling, Jacobi elliptic function, multiquadric curve, linear precision, extended complete Tchebycheff systems
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1 INTRODUCTION

We propose a new method to use Jacobi elliptic functions as blending functions for free-form curve formulations. For example, it is known that the shape of a rotating rope can be expressed by elliptic functions, and it is meaningful to use elliptic functions as blending functions. We follow the construction method of basis functions for Multiquadratic Curves: MQ-Curves, which is an Extended Complete Tchebycheff Systems.

2 MULTIQUADRATIC CURVS: MQ-CURVES

Multiquadratic(MQ-)curves [3] uses the space spanned by

$$U = \{1, t, \sqrt{c^2 + t^2}, \sqrt{c^2 + (1-t)^2}\}, \quad c \neq 0, t \in I = [0, 1]. \tag{1}$$

The normal curve $u \in A^3$ is an arc of an algebraic curve of order 4, since u lies in the intersection of the two hyperbolic cylinders $x_2^2 - c_1^2 = c^2$ and $x_3^2 - (1 - x_1)^2 = c^2$ [10]. Eck [3] presented a Bézier-like representation of MQ-curves and derived some interesting properties. We have now an easy approach to MQ-curves.

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2.1 Local Basis function

Based on Eck [3] in this subsection, we will think about MQ-segment. It is defined as

$$\phi_0(c,t) = a_0(1-t) + a_1\sqrt{c^2+(1-t)^2} + a_2\sqrt{c^2+t^2} + a_3 t \tag{2}$$

They introduced a complicated local blending functions for a MQ-segment with properties similar to the Bernstein basis in the case of polynomials. Eq(2) is written as follows

$$\phi(c,t) = \sum_{i=0}^{3} b_i \Omega_i(c,t) \quad t \in [0,1] \tag{3}$$

$\Omega_i(c,t)$ are defined

$$\begin{aligned}
\Omega_1(c,t) &= (s(c)+c)^2 \frac{\beta(c)}{\alpha(c)}(t - s(c)\sqrt{c^2+(1-t)^2} + c\sqrt{c^2+(t^2)}) \\
\Omega_2(c,t) &= \omega_1(c,1-t) \\
\Omega_3(c,t) &= t - \alpha(c)\Omega_1(c,t) + (\alpha(c)-1)\Omega_2(c,t) \\
\Omega_0(c,t) &= \Omega_(c,1-t)
\end{aligned} \tag{4}$$

where the auxiliary functions used in the above equations are defined as

$$\begin{aligned}
s(c) &= \sqrt{c^2+1} \\
\beta(c) &= \frac{s(c)}{s(c)+c} \\
\alpha(c) &= \frac{cs(c)}{2c^2+1+cs(c)}
\end{aligned} \tag{5}$$

In Eq.(3) the control points b_i is determined by

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & s(c) & c & 0 \\ 1-\alpha(c) & s(c) - \frac{\alpha(c)}{s(c)} & c & \alpha(c) \\ \alpha(c) & c & s(c) - \frac{\alpha(c)}{s(c)} & 1-\alpha(c) \\ 0 & c & s(c) & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{6}$$

The relations among the control points and Eq.(3) are

$$\begin{aligned}
b_0 &= \phi_0(c,0) \\
b_1 &= \phi_0(c,0) + \alpha(c)\phi_0'(c,0) = \phi_0(c,1) + (\alpha(c)-1)\phi_0'(c,1) + \alpha(c)\beta(c)\phi_0''(c,1) \\
b_2 &= \phi_0(c,1) - \alpha(c)\phi'(c,1) = \phi_0(c,0) + (1-\alpha(c))\phi_0'(c,0) + \alpha(c)\beta(c)\phi_0''(c,0) \\
b_3 &= \phi_0(c,1)
\end{aligned} \tag{7}$$

where ' denotes differentiation of $\phi(c,t)$ with respect to t. Note that $\sum_{i=0}^{3}\Omega_i(c,t) = 1$ for $t \in [0,1]$.

No explanation is done on how to derive $\Omega_i(c,t)$ in Eck [3]. Therefore we will derive these blending functions and extend them. From Eq.(3),

$$\begin{aligned}
b_1 - b_0 &= \alpha(c)\phi_0'(c,0) \\
b_3 - b2 &= \alpha(c)\phi_0'(c,0)
\end{aligned} \tag{8}$$

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The above equations means that the tangent vectors of a MQ-segment are parallel with $b_1 - b_0$ and $b_3 - b_2$ at the start point and end points just like the cubic Bézier curve. Eck [3] determine the coefficient of $\alpha(c)$ such that the curve in at Least three dimensions, b_1 is in fact the intersection points of the tangent in $\phi(c, 0)$ and the osculating plane in $\phi(c, 1)$, and vice versa for b_2.

From Eq(4),

$$\frac{\partial \Omega_0}{\partial t}(c, 1) = 0 \tag{9}$$
$$\frac{\partial \Omega_1}{\partial t}(c, 1) = 0 \tag{10}$$

Because of symmetry between $\Omega_0(c, t)$ and $\Omega_3(c, t)$ along $t = 1/2$ and $\Omega_1(c, t)$ and $\Omega_2(c, t)$

$$\frac{\partial \Omega_2}{\partial t}(c, 0) = 0 \tag{11}$$
$$\frac{\partial \Omega_3}{\partial t}(c, 0) = 0 \tag{12}$$

Furthermore the following equations are satisfied:

$$\begin{aligned} \frac{\partial \phi_0(c,t)}{\partial t}|_{t=0} = b_1 - b_0 \\ \frac{\partial \phi_0(c,t)}{\partial t}|_{t=1} = b_3 - b_2 \end{aligned} \tag{13}$$

The above conditions are rewritten by using a_i

$$\begin{aligned} b_1 - b_0 &= \frac{c\left(\sqrt{c^2+1}(a_3 - a_0) + a_1\right)}{c\left(\sqrt{c^2+1} + 2c\right) + 1} \\ &= \frac{c\sqrt{c^2+1}}{c\left(\sqrt{c^2+1} + 2c\right) + 1} \frac{\partial \phi(c,t)}{\partial t}|_{t=0} \\ b_3 - b_2 &= \frac{c\left(\sqrt{c^2+1}(a_3 - a_0) + a_2\right)}{c\left(\sqrt{c^2+1} + 2c\right) + 1} \\ &= \frac{c\sqrt{c^2+1}}{c\left(\sqrt{c^2+1} + 2c\right) + 1} \frac{\partial \phi(c,t)}{\partial t}|_{t=1} \end{aligned} \tag{14}$$

2.2 Derivation of Blending Functions

We assume that the blending functions $h_i(, ct)$ of the MS-segment are linear combinations of $f_0(c, t) = 1 - t$, $f_1(c, t) = \sqrt{c^2 + (1-t)^2}$, $f_2(c, t) = \sqrt{c^2 + t^2}$ and $f(_3(c, t) = t$. Furthermore we assume $h_2(c, t) = h_1(c, 1 - t)$, $h_3(c, t) = h_1(c, 1 - t)$.

Then $h_0(c, t)$ and $h_1(c, t)$ are defined by

$$\begin{aligned} h_0(c,t) = \sum_{i=0}^{3} a_i f_i(c,t) \\ h_1(c,t) = \sum_{i=0}^{3} b_i f_i(c,t) \end{aligned} \tag{15}$$

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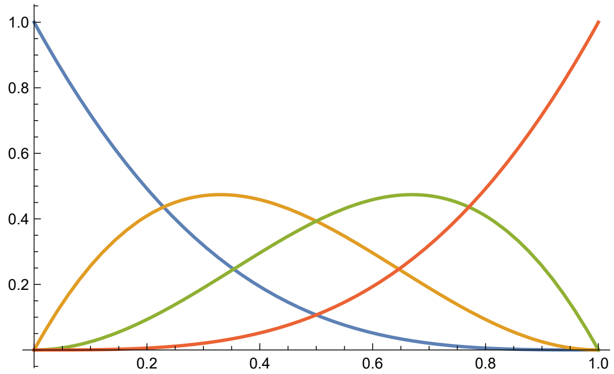

Figure 1: Blending functions of the MQ-segment, $c = 1$.

where a_i and b_i do not depend on either c or t and they are constants. Their number is equal to 8. The conditions on these functions are

$$
\begin{aligned}
h_0(c,0) &= 1 \\
h_0(c,1) &= 0 \\
\frac{\partial h_0(c,t)}{\partial t}|_{t=1} &= 0 \\
h_1(c,0) &= 0 \\
h_1(c,1) &= 0 \\
\frac{\partial h_0(c,t)}{\partial t}|_{t=0} &= -\frac{c\left(\sqrt{c^2+1}+2c\right)+1}{c\sqrt{c^2+1}}
\end{aligned} \tag{16}
$$

The last equation should be satisfied from Eq.(14). We need the following conditions for the partition of unity, which should not depend on either c or t and

$$
\begin{aligned}
a_0 + a_3 + b_0 + b_1 &= 1 \\
a_1 + a_2 + b_1 + b_2 &= 0
\end{aligned} \tag{17}
$$

We have 9 constraints and the number of variables is 8. Fortunately, these conditions are linearly dependent, and we can omit one of them. By solving a system of linear equations for a_i and b_i, $i = 0, 3$, we obtain $\Omega_i(c,t)$ in Eck [3].

Figure 1 shows $\Omega_i(c,t)$ with $c = 1$.

2.3 New Blending Functions

We can control the blending functions by changing the value $d = \frac{c\left(\sqrt{c^2+1}+2c\right)+1}{c\sqrt{c^2+1}}$ in Eq.(16). We will show blending functions with $d = 1/2$ and $d = 3$ in FIg.2.

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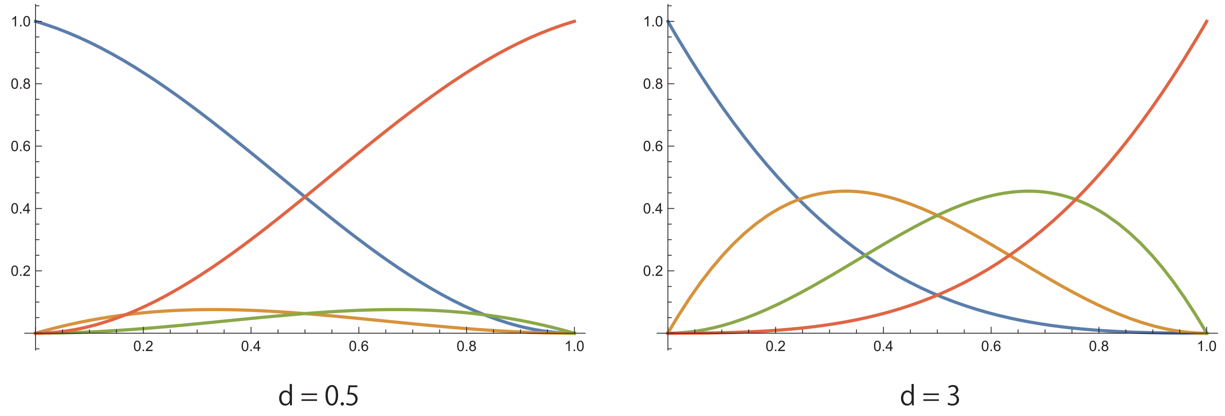

Figure 2: New blending functions of the MQ-segment, $(c, d) = (1, 1/2)$ and $(c, d) = (1, 3)$.

2.4 Linear Independence

The matrix representation M_0 of the above constrains is given by

$$\begin{pmatrix} 1 & \sqrt{c^2+1} & c & 0 & 0 & 0 & 0 & 0 \\ 0 & c & \sqrt{c^2+1} & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & \frac{1}{\sqrt{c^2+1}} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \sqrt{c^2+1} & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c & \sqrt{c^2+1} & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & \frac{1}{\sqrt{c^2+1}} & 1 \\ -1 & -\frac{1}{\sqrt{c^2+1}} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -d \\ 0 \end{pmatrix} \tag{18}$$

Note that the rank of M_0 is equal to 8, full rank, so we can add no linear constraint to the system. However, we need another constraint such that $a_0 + a_3 + b_0 + b3 = 1$ for the partition of unity of the blending

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functions. The solution of a_0, a_3 and b_0 and b_3 are given by

$$a_0 = -\frac{-8c^6 - 12c^4 - 4c^2 + \sqrt{c^2+1}c + 8\sqrt{c^2+1}c^5 + 8\sqrt{c^2+1}c^3}{\left(-c^2 + \sqrt{c^2+1}c - 1\right)\left(-8c^4 - 8c^2 + 4\sqrt{c^2+1}c + 8\sqrt{c^2+1}c^3 - 1\right)}$$

$$a_3 = -\frac{4c^5 + 3c^3 - \sqrt{c^2+1}c^2 - 4\sqrt{c^2+1}c^4}{-8c^4 - 8c^2 + 4\sqrt{c^2+1}c + 8\sqrt{c^2+1}c^3 - 1}$$

$$b_0 = 0$$

$$b_3 = -\frac{\left(\sqrt{c^2+1} - c\right)\left(-16c^8 - 36c^6 - 29c^4 - 10c^2 + 4\sqrt{c^2+1}c + 16\sqrt{c^2+1}c^7 + 28\sqrt{c^2+1}c^5 + 17\sqrt{c^2+1}c^3 - 1\right)}{c\left(2c^2 - 2\sqrt{c^2+1}c + 1\right)\left(c^2 - \sqrt{c^2+1}c + 1\right)\left(8c^4 + 8c^2 - 4\sqrt{c^2+1}c - 8\sqrt{c^2+1}c^3 + 1\right)}$$

As a matter of fact, from the above equations $a_0 + a_3 + b_0 + b_3 = 1$. Therefore, the blending functions satisfy the partion of unity. Another constraint which should be satisfied is

$$\frac{\partial \Omega_1(c,t)}{\partial t}|_{t=0} = d \tag{19}$$

This is satisfied because of the partition of unity as follows:

$$\frac{\partial \phi(c,t)}{\partial t} = \sum_{i=0}^{3} \frac{\partial \Omega_i(c,t)}{\partial t} = 0 \tag{20}$$

Hence

$$(\frac{\partial \Omega_0(c,t)}{\partial t} + \frac{\partial \Omega_1(c,t)}{\partial t})|_{t=0} = 0 \tag{21}$$

Then

$$\frac{\partial \Omega_1(c,t)}{\partial t})|_{t=0} = -\partial \Omega_1(c,t)\partial t)|_{t=0} = d \tag{22}$$

The constraint is automatically satisfied.

3 JACOBI ELLIPTIC FUNCTIONS

We can find many textbooks on Jacobi elliptic functions, for example, [11, 6] to know more about elliptic functions. We reviews the required background to understand the contents of this paper, based on Wikipedia [1] and Wolfram World [2]. The Jacobi elliptic functions are a set of basic elliptic functions. They are found in the description of the motion of a pendulum, as well as in the design of electronic elliptic filters. While trigonometric functions are defined with reference to a circle, the Jacobi elliptic functions are a generalization which refer to other conic sections, the ellipse in particular. The relation to trigonometric functions is contained in the notation, for example, by the matching notation sn for $\sin$.

The Jacobi elliptic functions are standard forms of elliptic functions. The three basic functions are denoted cn(u,k), dn(u,k), and sn(u,k), where k is known as the elliptic modulus. They arise from the inversion of the elliptic integral of the first kind,

$$u = F(\phi, k) = \int_0^{\phi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \tag{23}$$

where

$$\phi = F^{-1}(u,k) = \text{am}(u,k). \tag{24}$$

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From this, it follows that

$$\sin\phi = \sin(\mathrm{am}(u,k)) \tag{25}$$
$$= \mathrm{sn}(u,k) \tag{26}$$
$$\cos\phi = \cos(\mathrm{am}(u,k)) \tag{27}$$
$$= \mathrm{cn}(u,k) \tag{28}$$
$$\sqrt{1-k^2\sin^2\phi} = \sqrt{1-k^2\sin^2(\mathrm{am}(u,k))} \tag{29}$$
$$= \mathrm{dn}(u,k). \tag{30}$$

These functions are doubly periodic generalizations of the trigonometric functions satisfying

$$\mathrm{sn}(u,0) = \sin u \tag{31}$$
$$\mathrm{cn}(u,0) = \cos u \tag{32}$$
$$\mathrm{dn}(u,0) = 1. \tag{33}$$

The Jacobi amplitude ϕ is defined in terms of $\mathrm{sn}(u,k)$ by

$$y = \sin\phi = \mathrm{sn}(u,k). \tag{34}$$

The k argument is often suppressed for brevity so, for example, $\mathrm{sn}(u,k)$ can be written as $\mathrm{sn}u$.

The Jacobi elliptic functions are periodic in $K(k)$ and $K'(k)$ as

$$\mathrm{sn}(u+2mK+2niK',k) = (-1)^m\mathrm{sn}(u,k) \tag{35}$$
$$\mathrm{cn}(u+2mK+2niK',k) = (-1)^{m+n}\mathrm{cn}(u,k) \tag{36}$$
$$\mathrm{dn}(u+2mK+2niK',k) = (-1)^n\mathrm{dn}(u,k), \tag{37}$$

where $K(k)$ is the complete elliptic integral of the first kind, $K'(k) = K(k')$, and $k' = \sqrt{1-k^2}$ (Whittaker and Watson 1990, p. 503).

The standard Jacobi elliptic functions satisfy the identities

$$\mathrm{sn}^2u + \mathrm{cn}^2u = 1 \tag{38}$$
$$k^2\mathrm{sn}^2u + \mathrm{dn}^2u = 1 \tag{39}$$
$$k^2\mathrm{cn}^2u + k'^2 = \mathrm{dn}^2u \tag{40}$$
$$\mathrm{cn}^2u + k'^2\mathrm{sn}^2u = \mathrm{dn}^2u. \tag{41}$$

Special values include

$$\mathrm{cn}(0,k) = \mathrm{cn}(0) = 1 \tag{42}$$
$$\mathrm{cn}(K(k),k) = \mathrm{cn}(K(k)) = 0 \tag{43}$$
$$\mathrm{dn}(0,k) = dn(0) = 1 \tag{44}$$
$$\mathrm{dn}(K(k),k) = \mathrm{dn}(K(k)) = k', \tag{45}$$
$$\mathrm{sn}(0,k) = \mathrm{sn}(0) = 0 \tag{46}$$
$$\mathrm{sn}(K(k),k) = \mathrm{sn}(K(k)) = 1, \tag{47}$$

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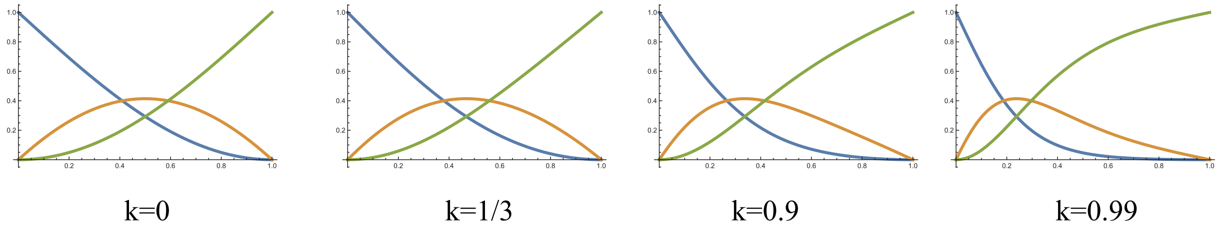

Figure 3: Basis functions with various k.

where K=K(k) is a complete elliptic integral of the first kind and $k' = \sqrt{1-k^2}$ is the complementary elliptic modulus (Whittaker and Watson 1990, pp. 498-499), and

$$\text{cn}(u,1) = \text{sech}\, u \tag{48}$$
$$\text{dn}(u,1) = \text{sech}\, u \tag{49}$$
$$\text{sn}(u,1) = \tanh u. \tag{50}$$

4 STANDARD BASIS FUNCTIONS CONSTRUCTION DEFINED WITH JACOBI ELLIPTIC FUNCTIONS

In this section, in order to make the differences clear with our basis function construction method, we describe standard construction of basis functions like Bernstein basis functions with Jacobi elliptic functions.

4.1 Asymmetrical Case

We pay attention to Eq.(38), Eq.(31) and Eq.(32) and similarly generalized trigonometric basis functions [7] we define the following basis functions:

$$u(t,k) = 1 - \text{sn}(K(k)t,k) \tag{51}$$
$$v(t,k) = \text{sn}(K(k)t,k) + \text{cn}(K(k)t,k) - 1 \tag{52}$$
$$w(t,k) = 1 - \text{cn}(K(k)t,k) \tag{53}$$

where $0 \le t \le 1$.

4.2 Symmetrical Case

As shown in Fig.3, the basis functions defined in the above are not symmetric along $t = 1/2$. We can modify them to be symmetrical without using $\text{cn}(K(k)t,k)$ as follows:

$$u_s(t,k) = 1 - \text{sn}(K(k)t,k) \tag{54}$$
$$w_s(t,k) = u_s((1-t),k) \tag{55}$$
$$v_s(t,k) = 1 - u_s(t,k) - w_s(t,k) \tag{56}$$
$$\tag{57}$$

where $0 \le t \le 1$. Figure 5 shows these basis functions with various k values.

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Because of Shape Uniqueness theorem [7, 8, 9], the shape of the rational quadratic Bézier curve with the second control point's weight $= \sqrt{2}$ is identical to the curves using Jacobi elliptic functions with any $k\,(0 \leq k < 1)$. although the blending functions are different as shown in Fig.(3).

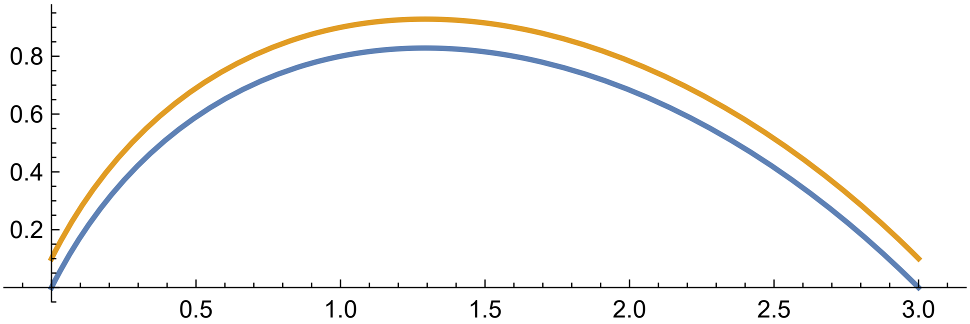
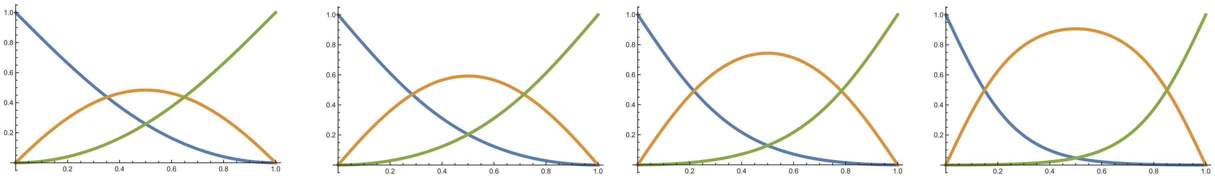


Figure 4: Rational Quadratic Bezier $w = \sqrt{2}$ with elliptic curve, which is shifted a little bit in the vertical direction. The control points are $(0,0)$,$(1,2)$ and $(3,0)$.



k=1/3 k=1/3 k=0.9 k=0.99

Figure 5: Symmetrical Basis functions with various k.

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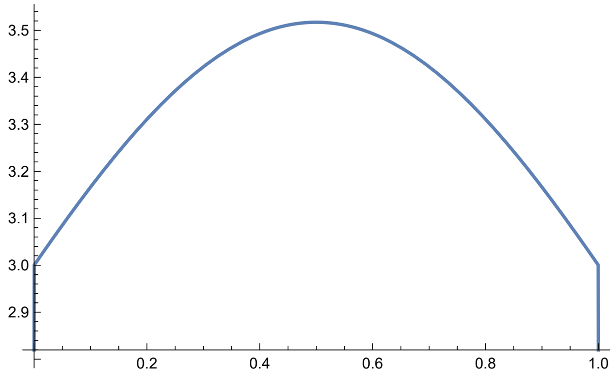

Figure 6: $v_s(t,k)^2/(u_s(t,k)w_s(t,k))$ with $k=1/3$.

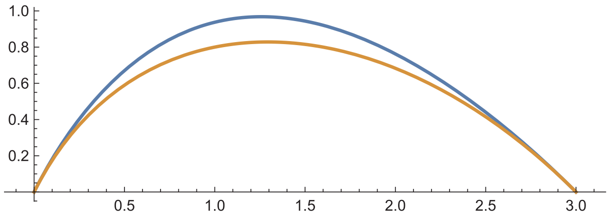

Figure 7: Comparison of the curves defined by asymmetrical and symmetric basis functions. The control points are $(0,0)$,$(1,2)$ and $(3,0)$.

However, for these basis functions is given by

$$\frac{v_s(t,k)^2}{u_s(t,k)\,v_s(t,k)} = \frac{(\mathsf{cd}(tK(k),k)+\mathsf{sn}(tK(k),k)-1)^2}{(\mathsf{cd}(tK(k),k)-1)(\mathsf{sn}(tK(k),k)-1)}. \tag{58}$$

The above formula is a function of t and not a constant as shown in Fig.6. Hence the curve defined by the symmetrical basis functions cannot be represented by a ration quadratic Bézier curve.

Figure 7 shows the curve in blue defined by asymmetrical basis functions and the curve in orange by symmetric ones with the same control points.

5 SHAPE REPRESENTED BY JACOBI ELLIPTIC FUNCTIONS

If we would like to use the graph of a specific function by a parametric curve, we need the following two properties:

1. One of the basis functions can represent the specific function.
2. The basis functions have the linear precision property.

In the next section, we will discuss about the linear precision property.

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5.1 Linear Precision

If all control points are spaced evenly on a straight line $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, and if the weights are unity, then the rational Bézier curve is in fact polynomial, tracing out the straight line in a linear fashion. This property is known as the linear precision property of Bézier curves [4, 5].

To represent a graph of Jacobi $\text{sn}(K(k)t, k)$, similar to the MQ-segment [3], we adopt $t-1$ and t as basis functions. Then the basis functions are

$$f_0(t,k) = 1 - t \tag{59}$$
$$f_1(t,k) = \text{sn}((1-t)K, k) \tag{60}$$
$$f_2(t,k) = \text{sn}(tK, k) \tag{61}$$
$$f_3(t,k) = t \tag{62}$$

Notice that the above basis functions do not satisfy the partition of unity, i.e. $\sum_{i=0}^{4} f_i(t,k) \neq 1$. A straight line segment can be represented by $\boldsymbol{P}_0 f_0(t,k) + \boldsymbol{P}_1 f_3(t,k)$. Using these basis functions, we define a new set of basis functions as

$$h_0(t,k) = \sum_{i=0}^{3} a_i f_i(t,k) \tag{63}$$
$$h_1(t,k) = \sum_{i=0}^{3} b_i f_i(t,k) \tag{64}$$
$$h_2(t,k) = \sum_{i=0}^{3} c_i f_i(t,k) \tag{65}$$
$$h_3(t,k) = \sum_{i=0}^{3} d_i f_i(t,k) \tag{66}$$

where a_i, b_i, c_i and d_i $(i = 0, \cdots, 3)$ are constants and they are determined by the constraints on the basis functions including the partition of unity. They can be determined by the similar method for the MQ-segment.

Here we can assume that $\sum_{i=0}^{3} h_i(t,k) = 1$. A curve $C(t,k)$ is defined with control points $\boldsymbol{Q}_i$ $(i = 0, \cdots, 3)$ as

$$\boldsymbol{C}(t,k) = \sum_{i=0}^{3} \boldsymbol{Q}_i h_i(t,k) \tag{67}$$

When we represent a scaled graph of $(P_{0x}(1-t) + P_{1x}t, (P_{1x} - P_{0x})\text{sn}(tK,k))$ with the scaling factor $(P_{1x} - P_{0x})$, the conditions on the control points $\boldsymbol{Q}_i = (Q_{ix}, Q_{iy})$ are given by

$$\sum_{i=0}^{3} \boldsymbol{x}_i h_i(t,k) = P_{0x}(1-t) + P_{1x}t \tag{68}$$
$$\sum_{i=0}^{3} \boldsymbol{y}_i h_i(t,k) = (P_{1x} - P_{0x})\text{sn}(tK,k)) \tag{69}$$

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From Eq.(68),

$$(a_0Q_{0x} + b_0Q_{1x} + c_0Q_{2x} + d_0Q_{3x}) = P_{0x} \quad (70)$$

$$(a_1Q_{0x} + b_1Q_{1x} + c_1Q_{2x} + d_1Q_{3x}) = 0 \quad (71)$$

$$(a_2Q_{0x} + b_2Q_{1x} + c_2Q_{2x} + d_2Q_{3x}) = 0 \quad (72)$$

$$(a_3Q_{0x} + b_3Q_{1x} + c_3Q_{2x} + d_3Q_{3x}) = P_{1x} \quad (73)$$

We have four variables Q_{ix} $(i = 0, \cdots 3)$ and four constraints expressed by the above equations. Generally we can solve the system of the above equations and we guarantee the linear precision property of the curve defined by with the basis functions. Note that when the partition of unity is guaranteed, the translation as well as rotation of the control points will preserves the shape of the curve.

6 BASIS FUNCTIONS BASED ON JACOBI ELLIPTIC FUNCTIONS

We are ready to derive basis functions using Jacobi Elliptic functions spaned by $(1-t, \mathsf{sn}((1-t)K, k), \mathsf{sn}(Kt, k), t)$ by making use of the procedures to construc basis function of MQ-curves. We would like to represent the shape of the rope by blending functions. Using $(t, \mathsf{sn}((1-t)K, k), \mathsf{sn}(Kt, k), t$ and adjusting parameters, the half shape of the rope is given by

$$\begin{aligned} C(t) &= (t, \mathsf{sn}(Kt, k)) \\ &= p_0(1-t) + p_1\mathsf{sn}((1-t)K, k) + p_2\mathsf{sn}(Kt, k), +p_3t \end{aligned} \quad (74)$$

where $p_0(0,0)$, $p_1 = (0,0)$, $p_2 = (0,1)$ and $p_3 = (1,0)$.

Then the matric M_3 similiar to M_0, $i = 0, 1, 2$ is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -K(k) & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -K(k) & 0 & 1 \\ -1 & 0 & K(k) & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -d \\ 0 \end{pmatrix} \quad (75)$$

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The solution is given by

$$\begin{aligned}
a_0 &= -\frac{d - K(k)^2 + K(k)}{(K(k)-2)K(k)} \\
a_1 &= -\frac{K(k) - d}{(K(k)-2)K(k)} \\
a_2 &= -\frac{dK(k) - d - K(k)}{(K(k)-2)K(k)} \\
a_3 &= -\frac{d(-K(k)) + d + K(k)}{(K(k)-2)K(k)} \\
b_0 &= \frac{d}{(K(k)-2)K(k)} \\
b_1 &= -\frac{d}{(K(k)-2)K(k)} \\
b_2 &= \frac{d(K(k)-1)}{(K(k)-2)K(k)} \\
b_3 &= -\frac{d(K(k)-1)}{(K(k)-2)K(k)}
\end{aligned} \tag{76}$$

Then the blending fcuntios $b_i(t)$ are defined by

$$\begin{aligned}
\begin{pmatrix} b_0(t) \\ b_1(t) \\ b_2(t) \\ b_3(t) \end{pmatrix} &= \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ b_3 & b_2 & b_1 & b_0 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} f_0(t) \\ f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} \\
&= M\,F
\end{aligned} \tag{77}$$

Therefore

$$\begin{aligned}
(q_0, q_1, q_2, q_3) \begin{pmatrix} b_0(t) \\ b_1(t) \\ b_2(t) \\ b_3(t) \end{pmatrix} &= (q_0, q_1, q_2, q_3) M\,F \\
&= (p_0, p_1, p_2, p_3) F
\end{aligned} \tag{78}$$

Then

$$(q_0, q_1, q_2, q_3) = (p_0, p_1, p_2, p_3) M^{-1} \tag{79}$$

For $(k, d) = (0.3, 2)$, we obtain $q_0 = (0, 0)$, $q_1 = (0.5, 0.856945)$, $q_2 = (0.5, 1)$ and $q_3 = (1, 1)$. Figure 8 show the locations of the control points and the curve generated with these control points to generate the half shape of the rope.

7 CONCLUSIONS

In this study, we propose to use Jacobi elliptic functions as blending functions for free-form curve formulations. We discussed in detail how to derive basis functions for Multiquadratic Curves: MQ-Curves, which is not

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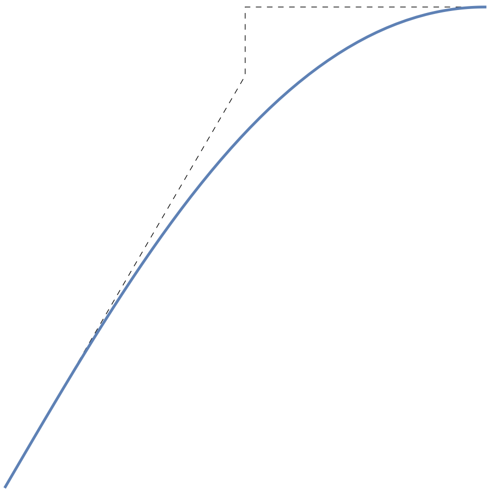

Figure 8: The half shape of the rope

explicitly explained in the original paper by Eck [3] In this paper, we focus on the usage of Jacobi elliptic functions, but our method to construct basis functions proposed in this paper is general and we can adopt other types of special functions as blending functions, that are our next research topic.

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