A Unified Approach for Airfoil Parameterization Using Bezier Curves

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Abstract. Parametrization is at the core of optimization, as it defines the design space that the optimizing algorithm explores. In addition, an industrial design workflow requires the components to be represented as the computer-aided design (CAD) model and seeks an optimized geometry also in the CAD format. This research aims to develop a unified approach for parameterizing two-dimensional airfoil geometries using the mathematical definitions of Bezier curves. In this work, an efficient and robust methodology is proposed which can be used to obtain the optimum locations of the Bezier control points, that approximately represents the airfoil shape (described using a set of data points). An error metric is defined to quantify the deviation of the approximated Bezier curve from the airfoil surface. The proposed methodology is validated on six different airfoil configurations, ranging from symmetrical NACA0012 airfoil to asymmetrical seven-digit NACA airfoil NACA747A315. This work is a step in the direction to create automated parameterization methodology, that can be implemented within a CAD system to parameterize three-dimensional aircraft wing models.

Keywords: Parameterization, CAD, Bezier curves, airfoil.
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1 INTRODUCTION

An Airfoil is the cross-sectional area of any wing, propeller blades, rotor and turbine blades which is mainly responsible for producing an aerodynamic force when it moves through a fluid. The shape of an airfoil mainly depends on the application for which they are employed and can be designed to be symmetric or asymmetric (about the centerline) in shape. An asymmetric airfoil is more efficient at producing lifts as compared to the symmetric airfoils. One of the bottlenecks for the optimization problems which involves computational fluid dynamics (CFD) is the high computational cost associated with each CFD analysis. Moreover, the total optimization time also depends upon the number of design variables employed for defining the geometry.

Parametrization is at the core of optimization, as it defines the design space that the optimizing algorithm explores. The success of any shape optimization methodology depends extensively on the type of parameterization technique employed [21]. One straightforward route which results in the most flexible parametrization strategy is to use the nodes of the computational mesh as the design variables. One major drawback for this parameterization strategy is that, as all surface mesh nodes can move independently, the appearance of non-smooth shapes can occur during the optimization process, which is not desirable, and would require implementation of smoothing algorithm.

In terms of airfoil design, the adopted parameterization strategy would ideally possess following characteristics:

- Able to represent a wide range of airfoil shapes.
- Smooth geometries with curvature continuity at the leading edge of the airfoil.
- Be computationally efficient in terms of runtime and memory usage for storing the parametrization.
- Able to produce CAD geometry which can be used for collaborative design approach.

A variety of airfoil parameterization techniques have been developed in the past: Hicks-Henne bumps [12], PARSEC method [23], Class Shape Transformation (CST) [16], B-Splines, Bezier-Bernstein polynomials etc. The Hicks-Henne bumps is parameterization strategy for two-dimensional airfoil in which the shape deformation is performed using analytical shape functions added onto the base airfoil. The functions are manipulated using the weight factors to provide a smooth deformation of the mesh surface. PARSEC method uses the analytical definitions of parameters defining the airfoil (leading edge radius, upper and lower crust position and curvature, trailing edge thickness and position etc.). All of these parameters have physical relevance which is why the PARSEC method is one of the most intuitive methods in parameterization. It is easier to control the maximum curvature on upper and lower surfaces and their location, but PARSEC method does not provide sufficient control over the trailing edge shape where important flow phenomenon can take place. It fits the smooth curve between the trailing edge and the point of maximum thickness which makes it difficult to optimize the design neat the trailing edge.

The parameterization methodologies discussed above can only be used for two-dimensional airfoil geometries. In this context, the Free-form deformation (FFD) techniques have been successfully implemented for aerodynamic shape optimization problems dealing with three-dimensional wing models [13], [15], [22]. The benefit of this approach is that it imparts smooth deformations to the analysis mesh and enables the parameterization to alter the thickness, sweep, twist, etc. for the design of an aerospace system. The CST parameterization can be used for two- and three-dimensional geometries, where the shape functions are defined using Bernstein polynomial and are manipulated by a series of identified class functions to be representative of base design. To the best of authors knowledge, these parameterization strategies use shape functions to describe the geometry, and it is not possible to directly obtain a suitable CAD model using these parameterization strategies. The mesh obtained from the optimization process must be translated into a CAD model before it can be used for further analysis or manufacturing assessments.

In an industrial design process, a part design typically starts with a CAD model and eventually have to deliver the optimized design in the CAD format. Hence, it desirable to directly use the CAD models for optimization to align well with the industrial ambition of having a more integrated design workflow. However, success of this methodology has been limited owing to several factors: (i) lack of efficient methodology to automatically parameterize CAD designs, (ii) no clear link between CAD model parameters and changes in these parameters effect model's performance, and (iii) inability to directly use the commercial CAD systems within optimization framework. In the recent past, some authors have attempted to work in this direction and developed CAD-based optimization processes involving CAD systems: B-splines [6], Beziers, non-uniform rational B-splines (NURBS) [7], [17], [29]. The limitations associated with the use of NURBS is that they do not work directly on the parametric CAD model created in a feature-based CAD system, thus the design intent and the parametric associativity captured in the choice of features used to build the model is lost. However, the former two approaches based on B-Splines and Beziers can be directly used to define the geometries within CAD systems (commercial or in-house). This has been shown in Ref. [3], [4], [20] where the parameters were created manually to define the Beziers and B-spline curves within commercial CAD system CATIA V5 and used within a gradient-based optimization framework.

The early works of Burgreen et al. [8-10] proposed the use of Bezier-Bernstein polynomials as an effective parameterization methodology for airfoils. Later, Venkataraman [26], used a fourth order and a cubic Bezier curve for constructing both the airfoil surface (top and bottom). Additional constraint was used to enforce slope continuity at the leading edge of airfoil and at the intersection of the two curves on the top and bottom surfaces. Sohn and Lee [24] presented a semi-analytical approach to determine the Bezier control points which can accurately approximate the upper and lower surface of the airfoil. In addition, they used Bezier control points to define the camber and then use the scale factor of the thickness variation as the design variables. All the control points were analytically obtained, except the first control point on either side of y-axis (found experimentally) which would impose C1 continuity at the leading edge. Balu and Selvakumar [5] used genetic algorithm to optimize the *x*- and *y*- coordinates of the Bezier control points (eight for both upper and lower surface) and applied the methodology on the RAE 2822 airfoil. Jaiswal [14] used the fourth-order, sixth-order and eighth-order Bezier curves to approximate the airfoil shapes by firstly using a linear least square functional problem to obtain initial set of Bezier control points and subsequently used them in a nonlinear least-square algorithm to obtain the final optimized control points. Wei et al. [28] used the basic Bezier parameterization approach to define the initial airfoil and subsequently used direct search algorithms to obtain the optimized control point locations to precisely fit the upper and lower surfaces of a low-Reynolds number airfoil.

The research mentioned above covered various research areas from aircrafts and enriched the application of Bezier curves for defining the shapes of the airfoils, which is of great significance to engineering application. But, as per the authors understanding there is no standard approach that can be used to obtain the parameterization based on Bezier control points for an aerodynamic wing surface. This paper is an effort in this direction and presents a standard methodology to obtain suitable Bezier control points, that can be used to fit an approximate curve on the airfoils data points defined in standard database [2] or the data points extracted from a section on the aircraft wing.

The remainder of the paper will first provide background of the Bezier curves, followed by the methodology to automatically parameterize the aerodynamic surfaces. Subsequently the results on six two-dimensional airfoil test cases (symmetric and asymmetric) will be presented and discussed. The paper will finish with the conclusions.

2 BEZIER CURVES

The Bezier curve is a parametric curve named after Pierre Bezier who developed and used them for designing the outer surfaces of Renault cars. The basic mathematical formulation of the Bezier curves consists of two parts:

- A set of points which define the geometry of the Bezier curve also known as control points.
- The order or the degree of Bezier curve is variable and is related to the number of points defining it: $n+1$ points define n^{th} degree curve.
- The basis function, also known as Bernstein polynomial.

The control points form the vertices of control polygon, which uniquely defines the shape of the curve as shown in Fig. 1. The curve passes through only the first and last control point, and the curve is tangent to the first and last polygon segment. Any point on the Bezier curve is parametrically defined as:

$$P(t) = \sum_{i=0}^{k} P_i B_{i,k}(t) \tag{2.1}$$

where, k corresponds to the degree of Bernstein polynomial and the value of t ranges from 0 to1. Further, the Bernstein polynomial can be described as:

$$B_{i,k}(t) = \binom{k}{i}(1-t)^{k-i}t^i \tag{2.2}$$

$i = 0,1,2,\dots,k$ where k is the degree of polynomial and

$$\binom{k}{i} = \frac{k!}{(k-i)!\,i!} \tag{2.3}$$

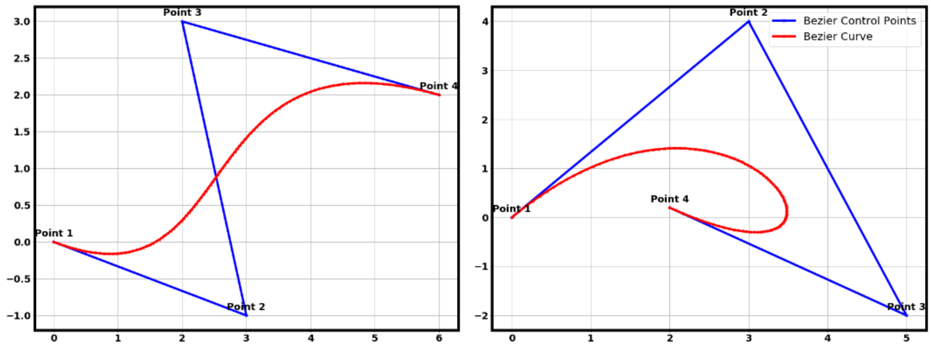


Figure 1: Bezier curve with control polygon.

The vector P_i represents the $k+1$ vertices of the control polygon and these vertices are further specified as control points. The Bezier curve need not pass through every control point, but they alter the shape of the Bezier curve by attracting the curve towards themselves. This gives a generalized control on the curve and it further aids in the optimization of the curve to fit complex geometries.

The following are some useful properties of Bezier curves [19], [30]:

- The basis functions are real.
- The degree of polynomial defining the curve is one less than the number of polygon vertices.
- The first and the last points of the curve are coincident with the first and last points of the defining polygons.
- The slope at the ends of the curve has the same direction as the polygon sides i.e. curve is tangent to the first and last segment of the polygon.
- The curve is contained within the convex hull of the defining polygon.
- The curve is invariant under an affine transformation.

In defining the airfoils usually either a composite Bezier curve (one for the upper surface and one for the lower surface is used) or a chained Bezier curve in which low order Bezier curves are joined together to form a complex shape as Bezier curve does not possess the property of locality since a change in a single control point changes the entire curve. Therefore, if a higher number of control points are needed for satisfactory description of a certain shape, a high-degree Bezier curve is generated. In addition to not possessing the property of locality, high-degree Bezier curves can also oscillate between control points as they are based on high-degree polynomials [27].

3 AUTOMATED PARAMETERIZATION METHODOLOGY

This work is focused on formulating automated parameterization strategy for two-dimensional airfoils, and subsequently extend it to parameterizing three-dimensional wing model. The motivation is to generate parameters which can be used within a CAD system to formulate a CAD-based optimization framework. This would be advantageous in terms of incorporating geometrical constraints, which are only possible within a CAD environment. These constraints may include the presence of structural elements like fuel-tank or spar or other structural elements.

In the field of computer graphics, the problem of fitting a parametric polynomial to approximate a set of data points have been tackled using error metric based on Least-squares [18]. An iterative approach using Newton-Raphson method was followed to obtain parametric values t for the cubic B-Spline polynomial that minimizes the distance between the data points and the polynomial curve. In this paper, a slightly different approach is followed to fit the Bezier curves on the data points defining an airfoil section.

3.1 Bezier Curve for Representing Airfoils

The methodology followed here employs two Bezier curves (one for upper and one for bottom surface) for defining the geometry of airfoil. The Bezier curve has a peculiar characteristic that it is always tangent to the first and last segment of the control polygon, and the first and last points on the curve are coincident with the first and last point of the polygon. These properties of Bezier curves are used here to obtain a constrained closed loop Bezier curve as discussed below:

The two Bezier curves (representing the upper and lower surface respectively) are constrained to have the same starting (leading edge) and ending point (trailing edge), this ascertains C0 continuity to obtain the closed loop curve. In addition, the two control points on either side of the leading edge are constrained to move only in the vertical direction to preserve the C1 continuity at the leading edge.

As discussed in section 2, to obtain a Bezier curve for representing airfoil surface, it is required to find a set of $n+1$ control points, P_i (where n is the degree of the curve), along with a set of parametric coordinates t, which minimizes the function

$$f = \left\| P_i B_{i,n}(t) - A_j \right\| \tag{3.1}$$

where, A_j represents the data points on the airfoil surface ($j = 1,2,3,\ldots,m$). The solution to eqn. 3.1 is not unique and values of P_i will be governed by the optimization algorithm. The parametric values t lies between $[0, 1]$ and are computed to be equally spaced within this interval dividing the entire curve into small sections. Using the values of t, the Bernstein polynomial of degree n is obtained, which along with the initial guess P_i^0 gives an ordered set of data points on the Bezier curve. These points on the Bezier curve are then added to an efficient data structure known as Kd-tree (K-dimensional tree). Kd-tree is a type of data structure that is used for efficient storage of information that is to be retrieved in subsequent searches. Two different Kd-tree are constructed: one for the upper surface and one for the lower surface. The Kd-tree facilitates efficient nearest neighbor search for a test point not in the tree to obtain closest points in the tree. For a data point on the airfoil, $\mathrm{D_A}$ (upper or lower), a query is performed with the respective Kd-tree to return nearest two points (X_1 and X_2) on the respective Bezier curve. The nearest distance of the point $\mathrm{D_A}$ from X_1 and X_2 is obtained by projecting the point (D_A) on the line-segment X_1X_2 and subsequently computing the perpendicular distance D_{Ai}. This distance D_{Ai} is computed for all the points on the airfoil (N), and subsequently used in a Least Square optimization framework defined as

$$Minimize\ f(x) = \sum_{i=0}^{N} (D_{Ai})^2 \tag{3.2}$$

The airfoils are characterized in terms of the chord length (c) which is the distance between the leading and trailing edge of the airfoil. To obtain the initial guess of Bezier control points, P_i^0 the chord length has been divided into $n-1$ equal segments (n is the degree of the Bezier curve), such that the x-coordinates of points $P_2, P_3, \ldots P_n$ are $c/(n-1)$, $2c/(n-1)$, … ,$(n-2)c/(n-1)$ respectively. The y-coordinates of the control points are computed from the maximum thickness of the airfoil. Apart from this, there are constraints imposed of following control points

- P_0: leading edge (0,0)
- P_{n+1}: trailing edge (c,0)
- P_1: vertical offset from leading edge (0,y_1)

With the initial guess of control points, an iterative optimization process is setup to minimize the distance function in eqn. 3.2 and obtain the optimum control points representing the airfoils.

3.1.1 Error metric

To quantify the difference between the approximated Bezier curve and the actual airfoil surface, an error metric is setup based on the value of D_{pi} computed at the last step of optimization. This signifies the error or deviation of the approximated parametric curve from the actual airfoil represented by a series of data points.

The basic flowchart of the proposed methodology is represented as below:

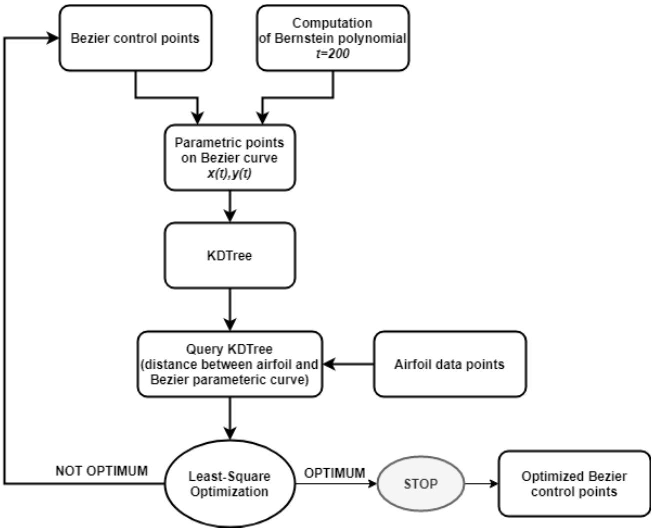


Figure 2: Flow chart of the airfoil Parameterization methodology.

4 RESULTS AND DISCUSSIONS

4.1 Order of Bezier Curve

One of the questions that arises here, what is the optimum degree of the Bezier curve that is sufficient to accurately represent the airfoil geometry? The degree of the Bezier curve is directly related to the number of points defining the control polygon, so increasing or decreasing the order of the Bezier curve can be achieved by changing the number of control points. In this section, a comparative study is performed to examine lower as well as higher degree Bezier curves to approximate one of the symmetrical airfoils.

4.1.1 NACA 0012

NACA 0012 is a symmetrical airfoil with zero camber and have been used as a benchmark problem for the AIAA CFD drag prediction workshop in 2016 [1]. The airfoil has maximum thickness at 12% of the chord from the leading edge. Here two Bezier curves are considered to approximate the upper and lower surface separately. A framework is formulated which can be used to generate any order of parametric Bezier curves, and the 4^{th}, 6^{th}, 8^{th}, 10^{th} and 12^{th} order are investigated here. The Bezier control points for different orders of curves are obtained using a least-square optimization methodology as described in section 3 to approximately fit the data points on the airfoil. The error metrics are plotted in Figure 3, where it can be seen that as the order of curve is increased from 6^{th} to 12^{th} order, the maximum error is reduced from 8e-5 to 5e-6. These errors can be computed in terms of % of the maximum thickness of the airfoil as 0.14% and 0.008% respectively. It is to be noted that as the order of Bezier curves increases, the number of control points also increases, which directly impacts the downstream optimization process applied to minimize or maximize certain objectives. So, there is a trade-off between the required accuracy and the computational cost. In **Figure 4**, it can be seen that the control points on the lower surface are the mirror image of the control points for upper surface, which is expected for the symmetrical airfoil. In addition, the deviation between the actual airfoil and approximated curve is small enough ($\approx 0.15\%$) for the 6th order Bezier curve. Therefore, in subsequent analysis 6th order Bezier curve is used to approximate the airfoil geometries.

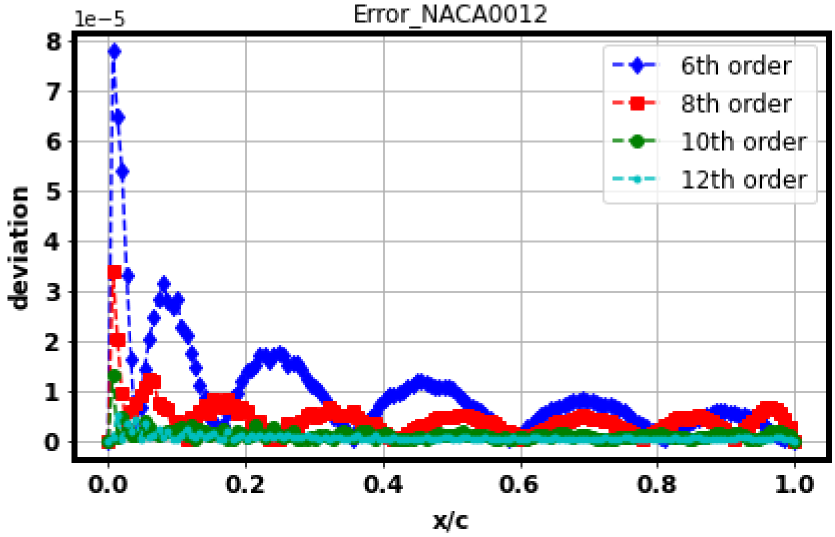


Figure 3: Error computations for various degrees of Bezier curves for approximating NACA 0012 airfoil.

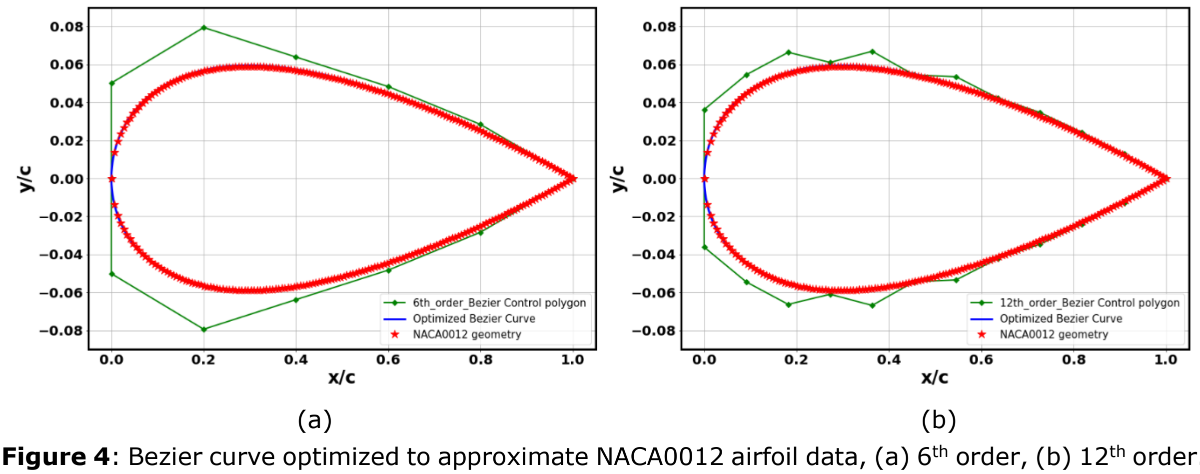


Figure 4: Bezier curve optimized to approximate NACA0012 airfoil data, (a) 6th order, (b) 12th order.

In order to demonstrate the robustness and applicability of the developed methodology, five other test cases have been used consisting one symmetrical and four asymmetrical airfoils.

4.2 Airfoil Approximations with 6th Order Bezier Curves

4.2.1 NACA66(4)-021

The second test case considered is the NACA 66(4)-021 airfoil which have been designed for laminar flow applications. NACA66(4)-021 represents a symmetrical airfoil with zero camber and maximum thickness of 21% at 45% chord. The control point locations of the Bezier curve are optimized using the developed methodology and the results are shown in **Figure 5**.

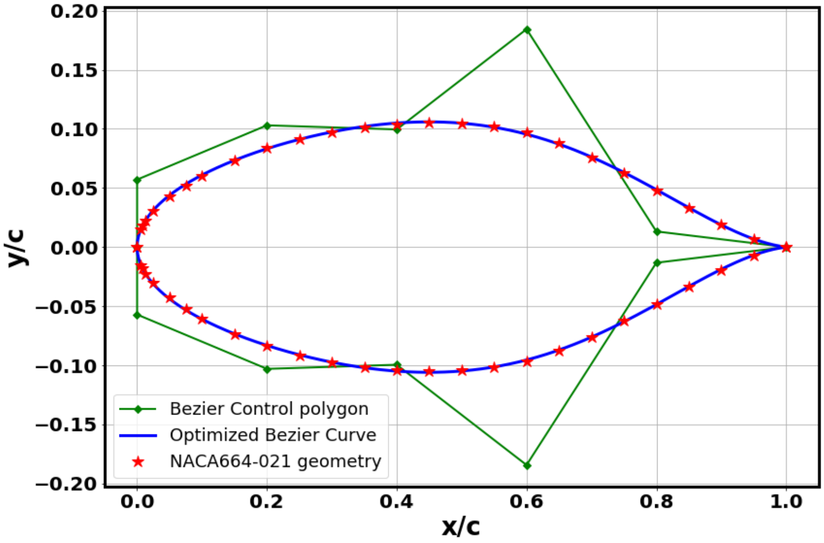


Figure 5: Optimized 6th order Bezier curve for NACA66(4)-021 airfoil.

4.2.2 RAE2822

The other test case considered is the asymmetric RAE2822 supercritical airfoil, which has been mainly analyzed in the transonic flow regime to model shockwaves in two-dimensional flows. Here the initial Bezier control points are chosen to be the same for the upper and lower surfaces, and subsequently optimized to fit the airfoil data points on both upper and lower surfaces, as shown in **Figure 6**.

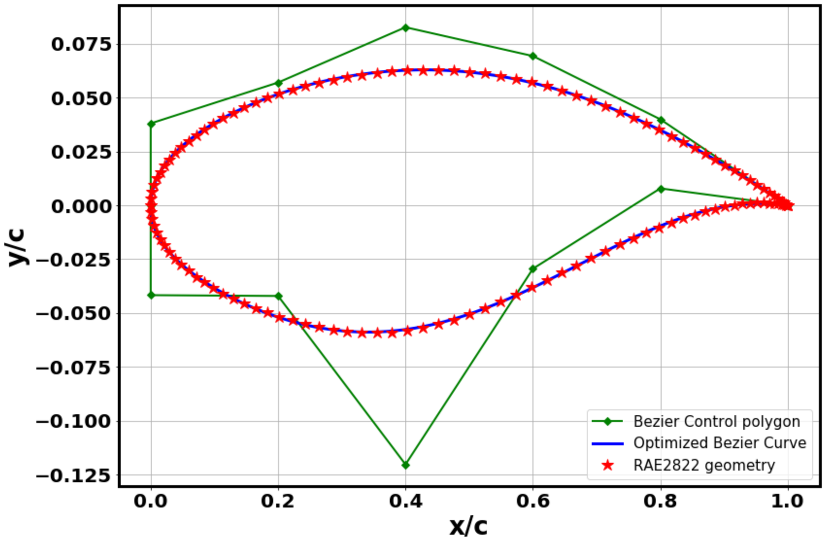


Figure 6: Optimized 6th order Bezier curve for RAE2822 airfoil.

4.2.3 Eppler airfoil

Eppler developed a series of airfoils by formulating accurate theoretical methods and employing the principles of inverse design. The Eppler airfoil E226 analyzed in this work has maximum thickness of 17.3% at 39.8% chord. The fitting of Bezier curve to approximate the E266 airfoil has been shown in **Figure 7**.

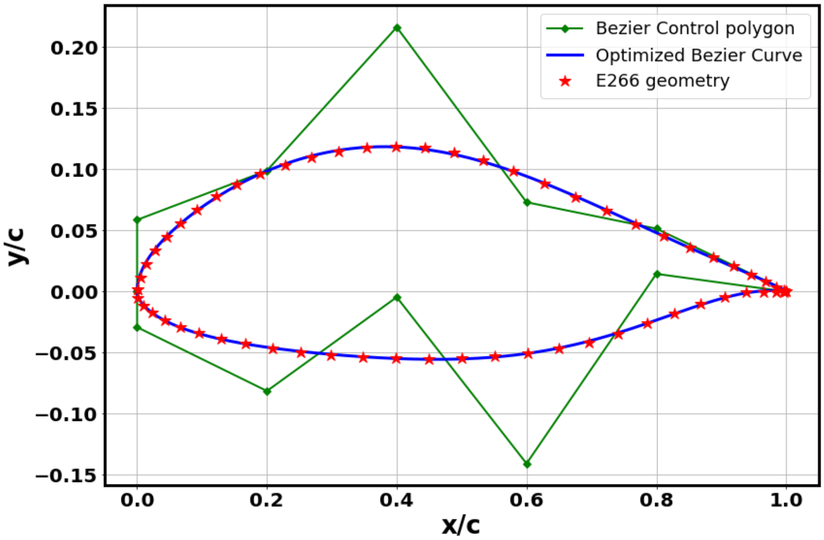


Figure 7: Optimized 6th order Bezier curve for E266 airfoil.

4.2.4 Helicopter Rotor Blade airfoil

The study of a new generation of helicopter blade airfoils was initiated by ONERA and AEROSPATIALE [25]. The aim of this study was to improve the performance of conventional airfoil profiles, such as NACA0012, for helicopter main rotor blade in hover and high speed. One of these airfoils is OA209 which has been used as one of the test cases in this study, and the results are shown in **Figure 8**.

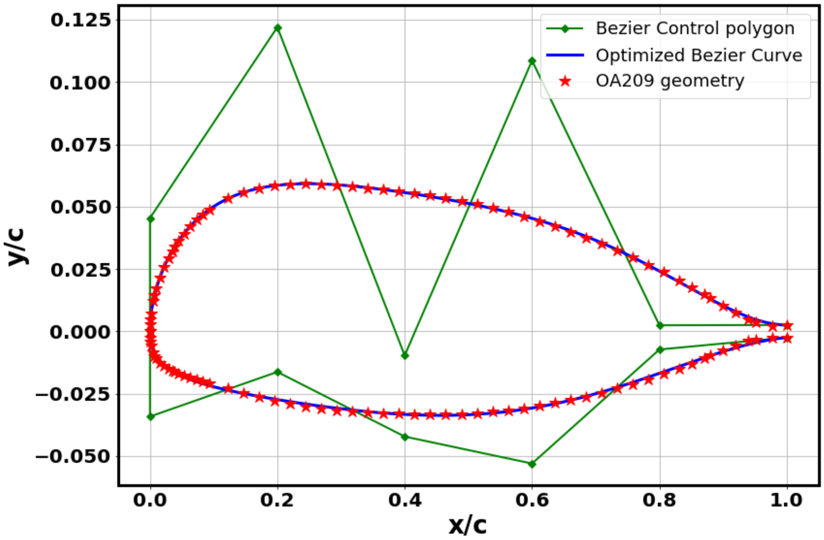


Figure 8: Optimized 6th order Bezier curve for OA209 airfoil.

4.2.5 NACA747A315

One of test cases in this study is the seven-digit NACA airfoil NACA747A315. This airfoil has a sharp cusp at the trailing edge, which makes it difficult to properly estimate the geometry near the trailing edge [11]. The Bezier curves are used to approximate the NACA747A315 airfoil, and the results are shown in Fig. 8.

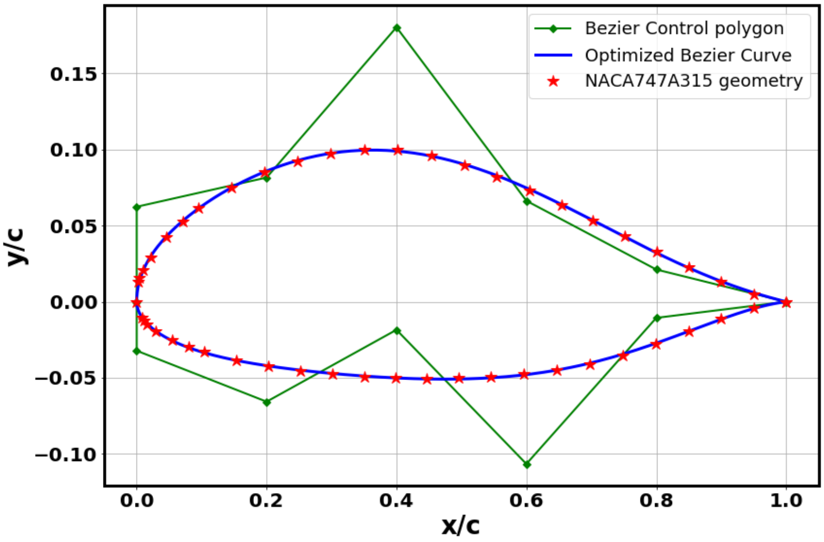
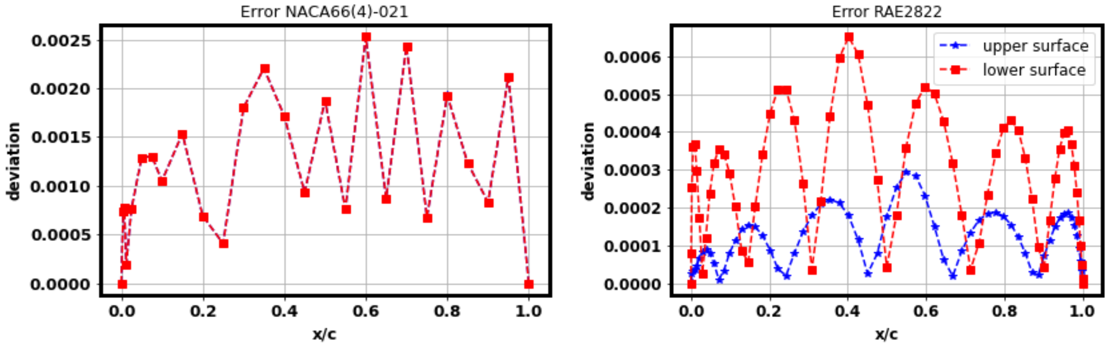


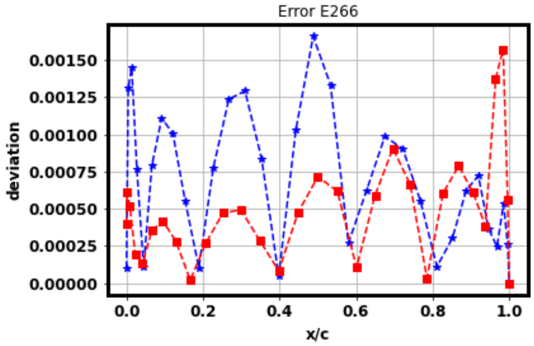
Figure 9: Optimized 6th order Bezier curve for NACA747A315 airfoil.

4.3 Error Metric

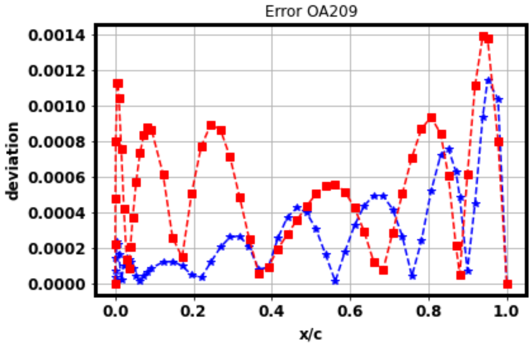
The error metric is computed for all the six airfoils investigated in this work and plotted in Fig. 9. It can be seen that for NACA0012 and RAE2822 airfoil geometries the maximum deviation is 6×10^{-5} and 7×10^{-4} respectively, while for the NACA66(4)-021, E266, OA209 and NACA746A315 the maximum deviation obtained are 2.5×10^{-3}, 1.6×10^{-3}, 1.4×10^{-3} and, 2.7×10^{-3}. These small values of maximum deviations, signifies that the methodology presented in this work is able to successfully parameterize the given set of airfoils using the Bezier curves.



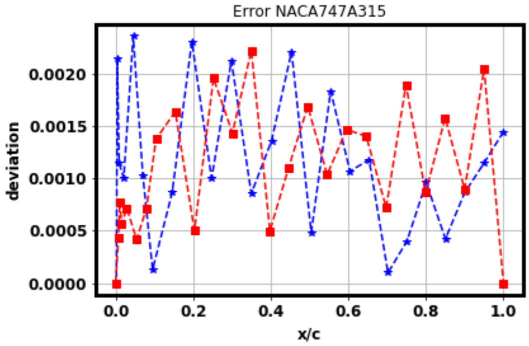
(a) (b)



(c)



(d)



(e)

Figure 9: Error metrics for different airfoil geometries, (a) NACA66(4)-021, (b) RAE2822, (c) E266, (d) OA209, and (e) NACA747A315.

5 CONCLUSIONS

In this paper, a unified approach is presented to obtain the locations of Bezier control points to fit the data points on the upper and lower surfaces of six different two-dimensional airfoil configurations. The developed methodology has shown promising results to approximate the airfoils with no noticeable visual difference between the estimated curves and the actual data points. The obtained Bezier control points can be used as the design variables within a gradient or gradient-free optimization framework to obtain the optimum airfoil configurations for varied applications. In addition, the availability of control points enables the creation of a faster and efficient approach to create large database (of varied airfoil configurations) for executing machine learning approaches. The methodology is configured to create Bezier curve (of any order) and optimize the position of control points to approximate an airfoil surface. This enables to approximate airfoil surfaces with higher degree of accuracy. As part of the future work, the developed methodology will be extended to parameterize three-dimensional wing models and formulating a framework to approximate the geometries using B-Splines.

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