Topology Optimization using Explicit Stress Tensor Analysis

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ABSTRACT

In this work, a framework for an inedited method for mechanical structures optimization is proposed. It is performed by re-arranging the Topology Optimization mesh obtained by BESO according to mechanical parameters. The principal stresses and the slope of the principal reference system are calculated, mesh elements are rotated, and a process of joining and size-modifying elements is performed. In a further step, a fine gradient based shape optimization may be applied. The main advantage of the method is that the final layout is created by modifying the orientation of the resulting elements, and an enhanced distribution of material is achieved. The goal is to overcome the sensitivity problems of other methodologies, and to reduce undesired checkerboard pattern. Finally, the preliminary results of a first implementation of the methodology are presented.

Keywords: Topology Optimization, FEM, Stress Tensor Analysis.
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1 INTRODUCTION

The goal of the Topology Optimization of a linearly elastic structure is the definition of the optimal material layout within a given domain. The structure is subjected to constrains and loads, which represent the boundary conditions of the problem. The desired result is the reduction of the solid volume compared with an initial design region.

The achievement of such task is relevant in many fields, and different approaches have been developed. These approaches include the solution of a constrained optimization [5]. The functions are minimized (or maximized) by the optimization method, which involves mechanical parameters such as compliance, eigenfrequency, and more. Due to the complexity of the problem, the achievement of high performances in terms of mass reduction is a complex task.

The well-known and widespread topological optimization (TO) methodologies are divided into two main categories: microscopic and macroscopic methods. Such distinction is substantial, because the result of the TO may be different due to the different strategy adopted.

In methods using a microscopic approach, such as SIMP [3] and BESO [12], the feasible domain available for the solid material is divided in a finite number of discrete elements. The density of the material of each element represents the design variables, which may vary with continuity in the SIMP method (density approach) or being binary in the BESO Method (evolutionary approach).

On the other hand, methods using macroscopic approaches, such as Level Set, describe the evolution of the boundaries of the solid domain in function of the design variables [15]. A discretization of the design domain is required as well, even if the implementation of this methodology does not intrinsically need the decomposition stage in a finite number of elements.

Many researches disclosed different variants of the said methodologies, highlighting their advantages and drawbacks. Eschenauer and Olhoff [9] carried out a comprehensive analysis of various aspects regarding the Topology Optimization. They proposed a review of the most widespread macroscopic and microscopic approaches. Methods adopting the microscopic paradigm were first presented by Bendsoe and Sigmund [4] [17]. Many efforts have been done in order to improve these methodologies. For instance, Christiansen et al. [7] combined Topology and Shape optimization in order to achieve more smooth geometries. Other efforts have been done even using genetic algorithms [24].

Another example is the work of Xia and Breitkopf [20], which is a derivation of the density approaches, such as SIMP, and deals with a multiscale topology optimization. A multilevel approach has been applied using BESO strategy at different optimization stages [6] as well.

Some works investigating the use of microscopic approaches highlighted some drawbacks. One of the most relevant issue is that SIMP, ESO, and BESO are able to give indication of the topology only in an implicit manner. For this reason, Guo et al. [11] proposed a TO strategy which explicitly controls the optimized layout with function representing geometrical features.

In general, macroscopic approaches, such as Shape Derivatives or Level Set [1] [23] [21] may be adopted as an alternative to microscopic approaches. Wang have investigated the application of level sets in [19], and the significant work to synthetize compliant mechanisms is detailed in [18]. To study the behavior of compliant mechanisms, a density-based topology optimization method (SIMP) has been adopted by Zhuo [22] as well.

Moreover, there are topic directly related to TO that have been investigated. As an example, the direct application to the CAD application using optimization strategies [8] [16].
Despite the large number of researches regarding the structural TO, the discussion about the choice of the best strategy to achieve the higher mass reduction is still an open topic. All approaches exhibit some advantages and are affected by some drawbacks. For instance, microscopic approaches are potentially able to produce a large number of topologies. On the other hand, they have a high sensitivity of the results to the choice of mesh resolution adopted to discretize the variable domain. Macroscopic approaches potentially are able to explicitly express the boundaries of the solid domain resulting by the optimization process. These approaches overcome the problem of undesired pixelization of the solid domain, typical for microscopic approaches. The main issue of this approach is the high dependency to the initial material configuration, and the risk in reaching a local minimum for the optimization problem.

The main goal of this paper is to present a novel methodology, which potentially combines the advantages of both microscopic and macroscopic approaches. Our proposal is avoiding the undesired checkerboard pattern affecting the result of microscopic approach TO. We introduce a novel method using an evolutionary microscopic TO approach in a first stage, combined with a refinement of the results by redefining the mesh. This second phase is carried out taking in account the results of the stress tensor analysis.

To better explain the main idea, Fig. 1. shows a simple example of a cantilever truss subjected to a vertical force. Fig. 1(a). depicts the result of the optimization of a truss, obtained using the BESO algorithm, which was implemented by using Abaqus script.

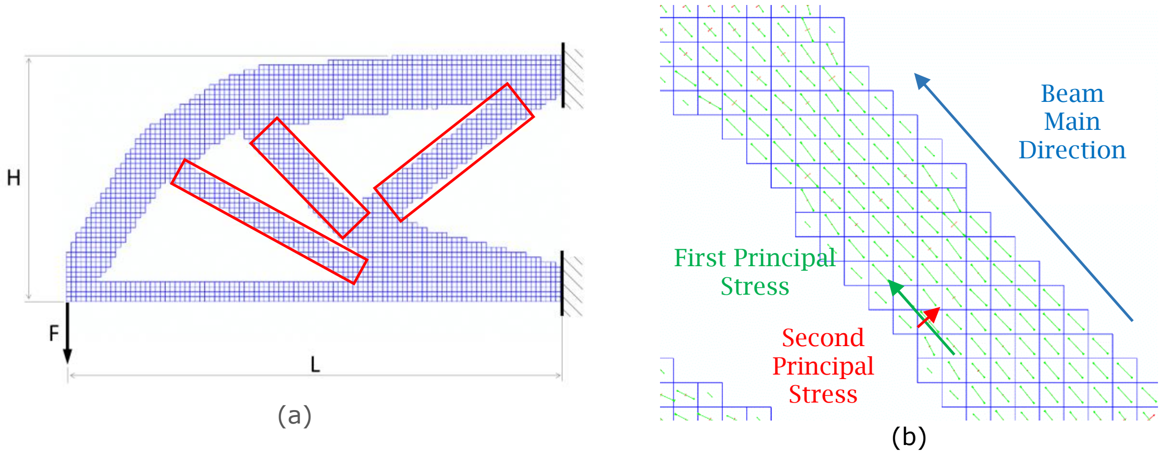


Figure 1: Boundary conditions for the case study and result of the process of Topological Optimization.

In this case, the boundary conditions are the application of a vertical force (F) in the left bottom vertex of the truss, and the constraint of its right side. In literature, this kind of procedures have been already largely studied, and the algorithms are well defined. To avoid tessellation of the resulting structure, the use of a sensitivity filter has been employed. The result of this combination is well-defined solid/void zones.

Some sub-structures are highlighted (red rectangles in Fig. 1(a).), which may be interpreted as beams. From a structural point of view, beam elements are the best in order to manage mono-axial states of tension, which means that the stress is a simple traction or a simple compression. In this example, a great number of elements composes such geometrical features. For this reason, it is important analyze the passage from a larger dimensional scale to a smaller one.

It can be observed that the beam sub-structures are not disposed in the same direction of the square elements they are composed of. This depends on a general initial definition of the mesh. In addition, analyzing the state of tension of the elements, depicted in Fig. 1(b)., the first principal tension is bigger than the second one, and has the same direction of the beam. This stress configuration may be interpreted as a pure compression for the beam. This observation is coherent with compression/traction state of the stress of the beam.

The implicit result of the TO is that the ideal layout includes some beam features at a macroscopic level. Conveniently, these beams have a mono axial state of stress. The same result may be explicitly obtained by observing (and averaging) the state of tension of the single elements which belong to the beams.

This parallelism between the macroscopic result of the TO and the analysis of the state of tension of its elements may support the foundation of the optimization methodology described in this research. The main hypothesis is that the geometric features resulting by a topology optimization, which may be recognized as beams, are subjected to simply traction or compression (and not subjected to bending, for instance). This means that an enhanced distribution of material occurs inside the workspace if the elements are purely compressed or in traction.

2 METHODOLOGY

The proposed methodology has been devised in order to incorporate information about the stress configuration in the definition of the mesh. This is coherently done with the empirical observations illustrated in Fig.1. It is composed of different phases, which may be implemented in a MATLAB code, for instance, by the use of the following routines: the Finite Method Analysis [2] [10] [13] and the BESO Topology Optimization [14] [25]. After obtaining the results from the previous steps, a re-meshing of the design space is performed. The re-meshing is based on the flux of tensions in the optimization in the design layout. To perform this stage, a routine for the rearrangement of the geometry has been developed.

Abaqus has been adopted to employ the whole process based on the following steps:

- discretization (mesh generation);
- rough BESO optimization;
- stress configuration evaluation;
- element rearrangement (rotation and connection);
- size (shape) optimization.

In the following subsections, each step is described.

2.1 Discretization

in the initial stage it is necessary to discretize the workspace defining a starting mesh. In this work, the design domain relates to a cantilever truss, and the elements are square Q4-linear elements. These elements are characterized by four nodes, which are the integration points, and linear shape functions.

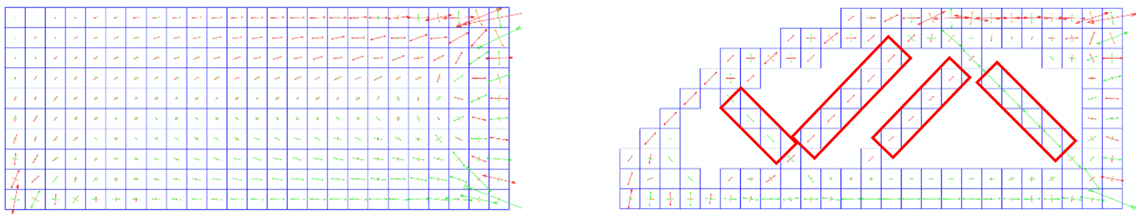
The physical quantities are evaluated at the center of each single element. Stresses and strains are computed, and the integral of the quadratic form representing the elastic strain energy is calculated as:

$$E_{el} = \frac{1}{2}\int_V K_{ijlm}\varepsilon_{ij}\,\varepsilon_{lm} \tag{2.1}$$

where K is the fourth order stiffness tensor of the element, and ε is the second order strain tensor.

In order to perform a comparison (for the check of the results of the MATLAB code) with the Abaqus software, the function built in function ESEDEN have been used. It has been done even because this physical quantity allows computing the sensitivity analysis to perform the optimization process.

Schematically, the discretization phase is depicted in Fig. 2. Fig.2(a). shows the initial analysis of the workspace AOI, highlighting the principal stresses, σ_I and σ_{II} in correspondence of the elements. These quantities are already expressed in the principal (local) system of reference. In addition to the original reference system of the element, the state of tension is described by all the components of the stress tensor σ_x, σ_y, and τ_{xy}.



(a) (b)

Figure 2: Stress fields for original and optimized structure. The analysis has been carried out using Abaqus while the optimization implementing the BESO algorithm.

2.2 Rough BESO optimization

The second phase consists in a rough Topology Optimization. The formulation of the problem may be stated as follows: the objective is the maximization of the stiffness of the structure (minimization of the total strain energy), and the volume becomes a fraction of the volume of the initial workspace.

Symbolically it may be written as follows:

$$\begin{cases} minimize: C = \frac{1}{2} K_{ijlm} \varepsilon_{ij} \varepsilon_{lm} \\ subject\ to: V_{Obj} - \sum_{1}^{N} V_i x_i = 0 \\ x_i = \begin{cases} 1 \\ x_{min} \end{cases} \quad 1 = 1, \dots, N \ (N\ number\ of\ elements) \end{cases} \tag{2.2}$$

The optimization is an iterative procedure, the Finite Element Analysis of the system is carried out, and the elastic strain energy corresponding to every single element is calculated again.

This phase is realized using a BESO method with the material interpolation scheme. If an element of the mesh discloses a sensitivity number, which is higher than a certain threshold, it is flagged as material element. On the other hand, if the sensitivity is lower, the element is flagged as void element. The sensitivity value for the material elements is equal to the elastic strain energy (Eqn. (2.1),), and for the void elements equal to zero (in correspondence of $x_i=x_{min}\approx 0$). The threshold value is set using a bisection algorithm, applied to the maximum and minimum sensitivity values. The elements flagged as material elements will be later redefined on the base of the principal stress configuration.

The result of the TO consists of the definition set of elements with high strain elastic energy. Low energy elements are firstly flagged, and a low density (and, consequently, a low Young modulus) is assigned. In a second phase, the elements with low density are erased. In Fig. 2(b)., the result of the TO is shown, where the sub-structures of the beams are identified. While investigating the nature of the stresses in such sub-structures, as depicted in Fig. 2(b)., it is highlighted that the major principal tensions are oriented in the same direction of the elements as indicated by the red rectangles.

2.3 Stress configuration evaluation

After the new set of elements have been defined, the next step is the computation of the state of tension for each element of the resulting mesh. A new finite element analysis of all the structure is performed, and each element is considered. A MATLAB function calculates the principal stresses and strains, and the correspondent slope for the principal directions.

This step is necessary because in the next stage, the material may be disposed along the stress flux, in order to reach an optimal distribution of matter and ensure better structural performances. This means that it is possible to formulate a rule to modify the starting mesh of the finite element analysis and better locally represent the structural response of the material aligning the elements to the principal directions. To do this, we evaluate the Mohr circle in the barycenter point of each element, which is defined by the traction/compression and shear tensions. Thanks to such analysis, it is possible to identify the rotation angle needed to impose to the finite element in order to orient it. The graphical interpretation of such procedure is schematically depicted in Fig. 3(b). The rotation angle of the generic element, as shown in Fig. 3(c)., is obtained by the use of known relations describing the Mohr circle for a plane state of tension:

$$\theta = \frac{1}{2} arctg\left(\frac{2\tau_{xy}}{\sigma_x + \sigma_y}\right) \tag{2.3}$$

This expression represents the transformation from the original local system (σ_x, σ_y, τ_{xy}) to the principal local system (σ_I, σ_{II}). By implementing this new definition, the new disposition of the element is obtained by a rigid rotation of the element itself by an angle θ.

2.4 Element rearrangement

In this phase, the elements are re-defined coherently with the results of the stress analysis, according to their mechanical properties. The goal is to acquire and consider new information regarding the orientation of the principal stress tensor directions to create a new mesh. For this purpose, the principal stresses and slope of the principal reference system are evaluated.

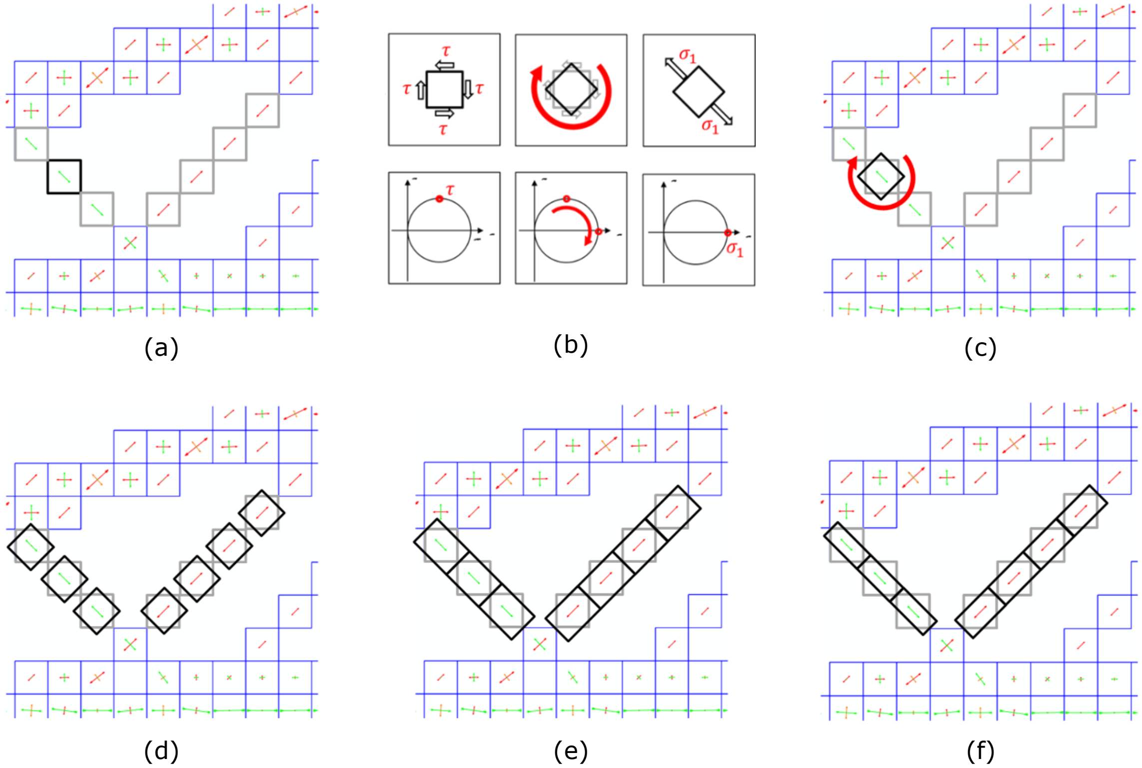


Figure 3: Re-orienting, connection and size optimization of the elements.

As depicted in Fig. 3(d)., all the elements are rotated coherently to the slope of their principal system of reference. Obviously, due to such rotation, it is not possible to preserve the continuity of the material. For this reason, the geometry of the mesh is recovered. This is done joining the adjacent elements by sharing the corresponding nodes, as shown in Fig. 3(e).

The rotation process of the elements means that the decoupling of the nodes is performed. The initial mesh is a standard square mesh and each node is shared by four elements. Usually the position of the nodes and the list of the nodes belonging to each element are stored in arrays. The rotation of the elements means that the vertices of the elements do not coincide anymore. For this reason, the array describing the nodes changes (the dimension is multiplied by four), and consequently the array of the elements changes as well (same dimension, different content). An example will be provided in the next section.

Furthermore, providing again the continuity means that adjacent elements will share the nodes after the rotation. Again, this is done modifying the nodes and elements arrays, decreasing this time the dimension of the nodes array.

2.5 Size Optimization

The last stage is a fine size (shape) optimization depicted in Fig 3(f). This is necessary because the re-definition of the mesh has an influence on the distribution of the stresses in the structure. In fact, the elastic strain energy stored in every rotated element may result different from the value corresponding to the last step of optimization.

For this reason, as depicted in Fig. 3(f)., a further step of optimization is realized. The main idea is that the section of the re-oriented elements depends on the module of the first (and unique) principal stress. Symbolically, this new step of rough BESO optimization may be carried out applying the relation:

$$C = E_{el} = \frac{1}{2}\int_V K_{ijlm}\varepsilon_{ij}\,\varepsilon_{lm} \tag{2.4}$$

where C is the sensitivity number of the last optimization step, or, in other words, the computed compliance of the element.

The goal of such procedures is the definition of the size of the elements so that their elastic strain energy density is the optimal for every element.

After the re-definition of the mesh, the elastic energy stored may be written for the mono axil state of tension of the elements:

$$C = E_{Opt} = \mathrm{V}(s_{Opt}) \cdot \frac{1}{2}\sigma\varepsilon \tag{2.5}$$

where S is the thickness of the element of the beam feature, and V is the volume (area) of the element itself.

Because some elements disclose a mono axial state of tension, it is possible to (locally) formulate the optimization problem using as optimization variables the geometric quantities is as follows:

$$\begin{cases} minimize\colon C = V(s_i) \cdot \frac{1}{2}\sigma\varepsilon \\ s_i = (0, s_{opt}) \;\; i = 1, \dots, M \; (M \; number \; of \; remaining \; elements) \end{cases} \tag{2.6}$$

This strategy is suitable for the elements having a pure traction/compression state of tension and characterizes the beam geometrical features. However, not all the zones of the structures have such mechanical behavior. For this reason, elements disclosing both principal tensions belong to the structure, providing connections between the beams sub-structures.

3 RESULTS

Some steps of the proposed methodology have been implemented in a program running in the MATLAB environment. No third parts packages are recalled in the code, and only basic logic, mathematical and graphical features have been used.

First, the state of tension in the domain has been evaluated. These results are shown in Fig. 4(a)., which portrays the state of tension of the truss. For each single element the vectors representing σ_x(blue vectors), σ_y,(magenta vectors) and τ_{xy} (green vectors) are visualized. On the other hand, Fig. 4(b). shows the same elements rotated coherently to the principal directions. The principal tensions σ_I (green vectors) and σ_{II} (red vectors), are shown as well. Moreover, in Fig. 4(c). the result of the application of a BESO TO algorithm is shown. Fig. 4(d). depicts the effect of the rotation along the principal directions applied to the survived elements.

Finally, Fig. 4(e). portrays the result of the rotation and re-arrangement of the elements of the mesh. This is done coherently to the slope of the principal directions, and to the elastic strain energy, evaluated in each single element of the mesh shown in Fig. 4(d). The result is that the new geometry is less affected by the checkerboard appearance than the one shown in Fig 4(c). In other words, the proposed method is able to produce a great variety of topologies, typical of microscopic approaches, with a good description in terms of continuity and smoothness of the boundaries, typical of macroscopic approaches.

At present the code is still under refinement, and the results in terms of definition of the mesh and boundaries are still under development. Anyway, we have been able to proceed with preliminary tests.

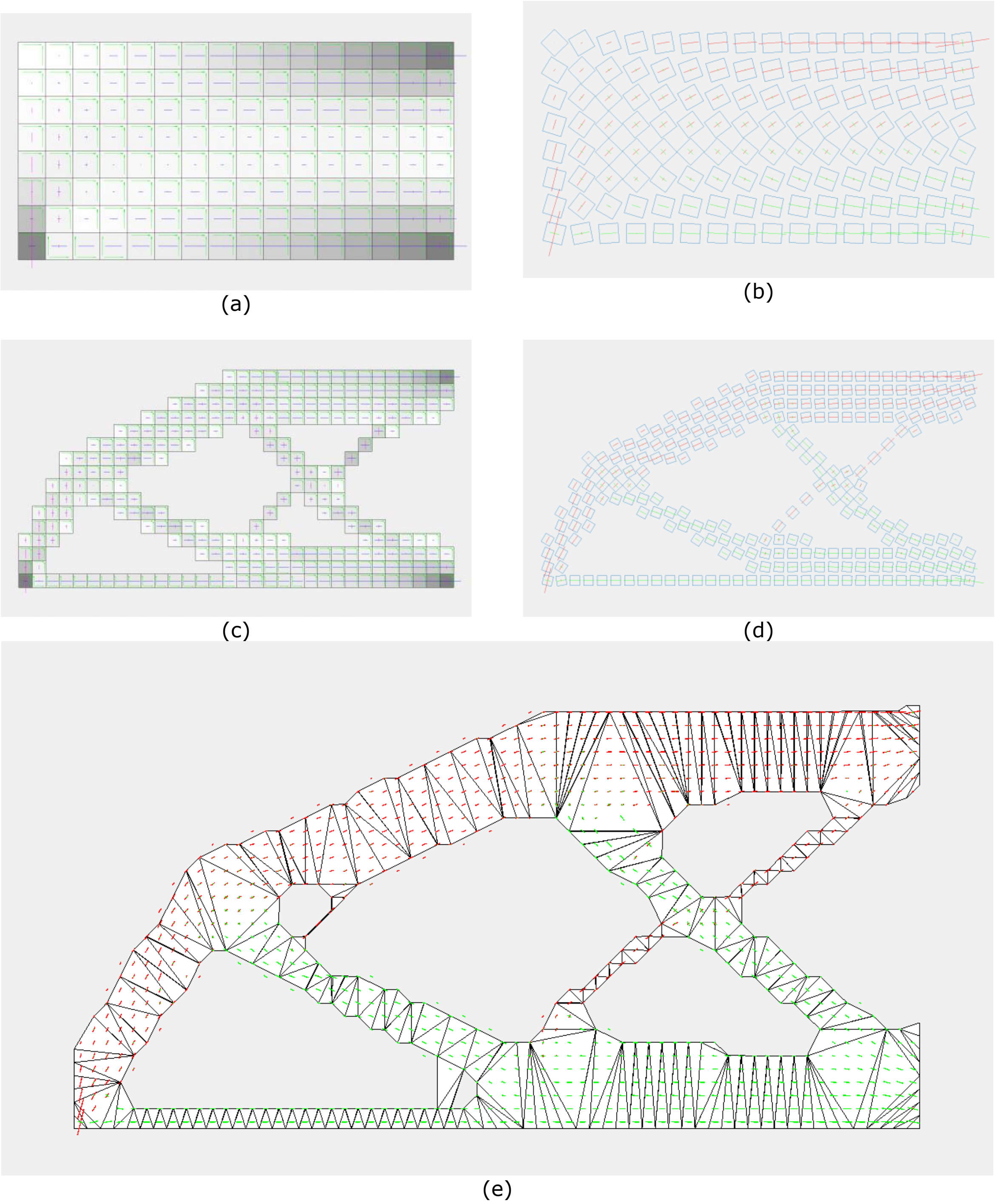


Figure 4: MATLAB implementation: (a) and (d) are the results of the FEA on the design space; (c) and (d) depict the FEA analysis carried out on the result of the BESO TO procedure; (e) shows the result of the proposed approach.

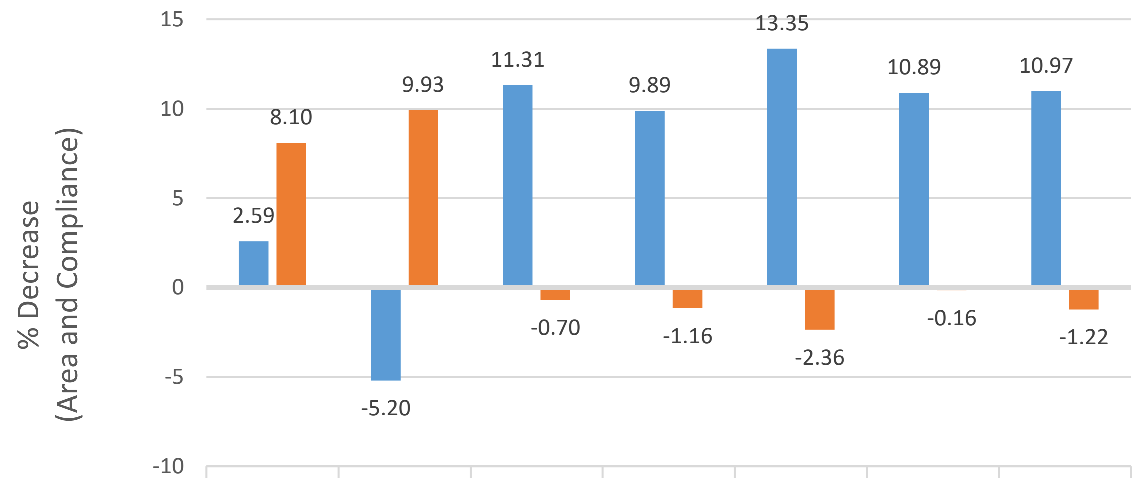
3.1 Comparison between BESO and proposed approach

In this section we present the first results of the comparison between the rough BESO TO and the proposed methodology. The test has been done considering the boundary conditions specified in Fig. 1(a). (H=100mm, L=200mm, F=1000N). Moreover, several repetitions have been performed with different grids for defining the starting mesh. The design space has been discretized in a grid of 16x8 elements, then 20x10 elements, etc., as reported in Fig. 5. More specifically, we refer to Fig. 4(c). and Fig. 4 (e)., depicting the results of the BESO method, and the proposed method respectively. In both cases, we evaluated the total area (volume) and the compliance. In order to compare such results, we defined two indexes:

$$\% \, Area \; decrease = \frac{(\text{Area}_{BESO} - \text{Area}_{Re-Mesh})}{\text{Area}_{BESO}} \tag{3.1}$$

$$\% \, Compliance \; decrease = \frac{(\text{Compliance}_{BESO} - \text{Compliance}_{Re-Mesh})}{\text{Compliance}_{BESO}} \tag{3.2}$$

Eqn. (3.1). and (3.2). describe the indexes, which represent the improvement of the new method (Re-Mesh) in terms of area and compliance decrease, compared to a standard BESO. The results of the comparison for different elements sizes are depicted in Fig.5.



	16 x 8	20 x 10	32 x 16	40 x 20	50 x 25	64 x 32	80 x 40
% Area Dec.	2.589	-5.202	11.314	9.890	13.353	10.885	10.972
% Compl. Dec.	8.095	9.927	-0.702	-1.158	-2.362	-0.157	-1.223

Figure 5: Results of the comparison between the BESO TO, and the propose methodology.

It can be noticed that most of the results are positive, which means that the proposed methodology gives better results. When a value is negative, for example we have an increase of the area, on the other hand we have a higher decrease (in percent) of the compliance, which means that globally there is a more efficient use of material.

3.2 Results discussion and further developments

The first tests are promising; however, there are still some issues to deal with. In fact, generating the new topology, there is still dependence of the result on the initial size of the mesh elements. Even if it introduces some advantages in terms of decrease of area and compliance, the re-definition of the mesh at the moment cannot completely modify the topological class of the structure.

Moreover, even if the tessellation of the topology is strongly reduced, a smooth perimeter is far from being obtained. The results of the proposed methodology may be further refined using parametric curves for the description of the boundary of the structure itself.

Compared to the use of the boundaries of a set of discrete elements, parametric curves are more feasible to provide a description of the solid domain. The reason is that the output of the methodology is improved, compared to the appearance of topology in microstructural TO, avoiding tessellation. Moreover, if the boundary of the structure is explicitly described by parameters, it is possible to apply a gradient based shape optimization method.
For this purpose, we propose the use of Non-Uniform Rational B-Spline (NURBS) curves. NURBS curves offer continuity and flexibility for the analytic and parametric representation of TO results. More in detail, their formalization provides three sets of parameters, which represent the three different degree of freedom, which rule a new curve: control vertices, weights corresponding to each control vertex and values of the knot vector.

Fig. 6. shows how the nodes of the rotated and re-joined elements are used to construct the NURBS curve. Nodes are used for defining the polygon vertices, being a partial input for calculating NURBS curves and the sequence of vertices is converted into the set of control points. The NURBS curves defined in this way may incorporate line and curves. In the present work we suppose to define the control points coordinates and eventually optimize the weights. Adopting this strategy, a good trade-off between complexity (number of parameters) and the possibility represent complex shapes may be achieved. This is done in order to define the boundaries of the domain of the solid material. If different beams are adjacent, the NURBS curves provide the necessary continuity.

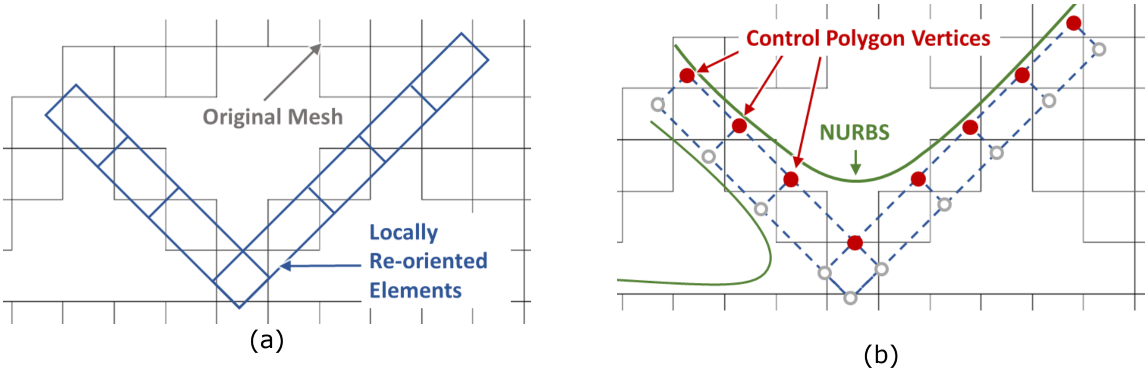


Figure 6: NURBS application: (a) re-oriented elements, (b) definition of fitting NURBS.

4 CONCLUSIONS

This work proposes a conceptual framework for a novel TO approach. It is performed by re-arranging the TO mesh obtained by BESO according to mechanical parameters. The principal stresses and the slope of the principal reference system, are calculated, rotated, and a process of joining and size-modifying elements is performed.

The main advantage of the method is that the final layout is created by modifying the orientation of the resulting elements, so that the discretization of the Area of Interest (AOI) is more accurately represented by the distribution of tension. This is important because, the typical fuzzy aspect of the microstructural approaches is avoided. The framework has been implemented in a MATLAB program, some tests have been done, and the results confirm the improvement of the efficiency in the use of the material (less material, less compliance).

Future research includes parametric representations of the resulting topologies using NURBS curves to provide a smooth contour.

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