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On a blossom-based approach for interpolating non-iso-parametric curves by B-spline surfaces

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ABSTRACT
Interpolating non-iso-parametric B-spline curves by B-spline surfaces remains a challenging problem in Computer-Aided Geometric Design. The solution typically involves composition and product of B-spline bases, leading to a system of linear equations. In this paper, we discuss the interpolation of such curves using the representation of B-splines in polar form, or Blossom, as proposed in matching trimmed surfaces. Although the discussion is limited to curves which are linearly mapped in the corresponding parameter domain, the interpolation of other types of curves could be inspired.

KEYWORDS
B-spline surfaces; B-spline curves; Non-iso-parametric curves; Curve interpolation; Polygonal complexes; Blossoming

1. Introduction

Curve interpolation is a problem that has frequently been visited by many authors [2, 12, 13]. The efficacy of the adopted solutions nearly removed the halo of difficulty the problem used to be assuming. However, those solutions rely in a fundamental way on the assumption that the interpolated curve is iso-parametric along one or the other of the parameters controlling the interpolating surface. In fact, as soon as this assumption is removed, the level of difficulty of the interpolation problem rises enormously.

Ferguson and Grandine [4] are perhaps the first to aim for the construction of B-spline surfaces interpolating non-iso-parametric curves. However, to our knowledge, there was never a follow up to that paper. Most of related research may rather be found in the area of curves on surfaces [16], using blossoming techniques [15], and also in the area of interpolation of arbitrary networks of curves [9].

However, the work of Hu&Sun [6] is the most directly relevant to the research reported in this paper. Especially so, since the technique proposed here for the solution of the interpolation of non-iso-parametric curves is a slight adaptation of the one reported in that paper and used there for trimmed surface matching.

In the design of curves on surfaces, the interest is focused on a curve lying on the surface. This may be obtained by mapping a curve lying in the parameter domain onto the 3D surface.

Drawing a curve on the surface is rather simple, as it can be mapped point by point following the parameter line in the parameter domain onto the 3D surface. However, plenty of research has been conducted on the specification of such curves in control point representation [16], which is not straightforward.

Our interest in curves on surfaces here is just to provide a point of focus as to which parts of the surface would need to be modified in order to establish interpolation.

This paper reports work that is still in progress. It is structured as follows: Section 2 provides preliminary definitions of B-spline curves and surfaces. This shows the simplicity of obtaining interpolation of points by B-spline curves on the sole basis of those definitions.

Section 3 shows how simple and direct it is to obtain the interpolation of B-spline iso-parametric curves by B-spline surfaces using the notion of B-spline polygonal complexes [1, 11]. By comparison, Section 4 resorts to the use of polar form or blossom in order to interpolate B-spline non-iso-parametric curves. The final section of the paper concludes with a conclusion and some suggestions for further work.

2. Preliminaries

This section shows how the basic information on B-spline curves and surfaces can naturally lead to the notion of B-spline polygonal complexes [1, 11] and therefore to curve interpolation. Furthermore, it also shed light on how easy it is to interpolate iso-parametric curves.

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2.1. B-spline curves

Given a sequence of control points $[p_0, p_1, \ldots, p_m]$, a non-decreasing sequence of knots $[t_0, t_1, \ldots, t_n]$, and a parameter $t \in [t_p..t_{m+1}]$, a B-spline curve of degree d (such that $m = n - d - 1$) is defined as follows:

$$C(t) \equiv \sum_{i=0}^{m} N_i^d(t) p_i \tag{1}$$

where N_i^d is the i^{th} B-spline basis function of degree d.

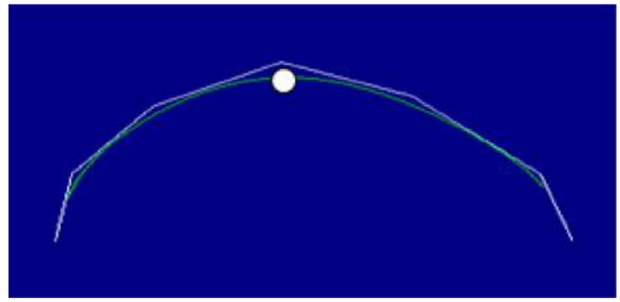

Figure 1. Interpolating a point corresponding to parameter t_k by a curve.

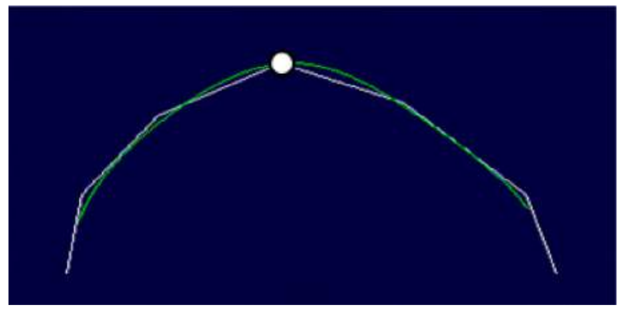

Figure 2. Curve altered to interpolate the control point p_{k-2}.

2.2. Point interpolation

Local effects are among the properties of the B-spline basis function N_i^d. In fact, at most $d + 1$ control points affect the curve $C(t)$ at any selected parameter t. These have the consecutive indices ranging from $i - d$ to i, where $t_i \leq t < t_{i+1}$.

2.2.1. Interpolating a point corresponding to an arbitrarily selected parameter

In general, for any arbitrarily selected parameter t, the summation of $C(t)$ in Eqn. (1) will have $d + 1$ non-zero terms, which evaluates to a point (here $t_k \leq t < t_{k+1}$). Thus, for example, in the cubic case (i.e., when $d = 3$), this point would be:

$$p'_{k-2} = p_{k-3} N_{k-3}^3(t) + p_{k-2} N_{k-2}^3(t) + p_{k-1} N_{k-1}^3(t) + p_k N_k^3(t) \tag{2}$$

which is obviously interpolated by the curve $C(t)$. Conversely, if the point p_{k-2} is replaced by the point:

$$\frac{1}{N_{k-2}^3(t)} (-p_{k-3} N_{k-3}^3(t) + p_{k-2} - p_{k-1} N_{k-1}^3(t) - p_k N_k^3(t)) \tag{3}$$

in the control point sequence $[p_0, p_1, \ldots, p_m]$, the resulting curve would interpolate p_{k-2} itself.

As shown in [2], there are infinitely many ways of obtaining such interpolating effects. But, the ease with which these effects are obtained is mainly due to having, as the subject of primary focus, a single parameter within the range of the initial knot sequence.

2.2.2. When the selected parameter is one of the curve knots

However, when t is selected as one of the knots $[t_0, t_1, \ldots, t_n]$, the number of control points affecting the curve $C(t)$ at parameter t reduces to d (by dismissing point p_i). Thus, for example, in the cubic case (i.e., when $d = 3$), for any one t_k of those knots, the summation of $C(t_k)$ in Eqn. (1) reduces to a point:

$$p'_{k-2} = p_{k-3} N_{k-3}^3(t_k) + p_{k-2} N_{k-2}^3(t_k) + p_{k-1} N_{k-1}^3(t_k) \tag{4}$$

which is obviously interpolated by the curve $C(t)$ (see Fig. 1).

Conversely, if point p_{k-2} is replaced by the point:

$$\frac{1}{N_{k-2}^3(t_k)} (-p_{k-3} N_{k-3}^3(t_k) + p_{k-2} - p_{k-1} N_{k-1}^3(t_k)) \tag{5}$$

in the control point sequence $[p_0, p_1, \ldots, p_m]$, the resulting curve would interpolate p_{k-2} itself (see Fig. 2). The act of replacing one control point by another to make interpolating this point possible is referred to as control point *repositioning* [1].

2.3. B-spline surfaces

Given the following list of items:

- a grid of control points p_{ij}, where $0 \leq i \leq m_1$ and $0 \leq j \leq m_2$;
- a knot vector in the u direction: $[u_0, u_1, \ldots, u_{n_1}]$;
- a knot vector in the v direction: $[v_0, v_1, \ldots, v_{n_2}]$;
- a degree d_1 in the u direction ($m_1 = n_1 - d_1 - 1$); and
- a degree d_2 in the v direction ($m_2 = n_2 - d_2 - 1$);

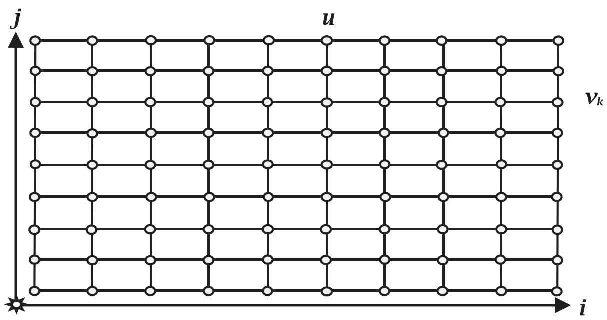

Figure 3. Control points grid of a B-spline surface.

A B-spline surface (see Fig. 3) is defined by the following fundamental expression:

$$S(u,v) \equiv \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} N_i^{d_1}(u)N_j^{d_2}(v)p_{ij} \tag{6}$$

where $u \in [u_{d_1}..u_{m_1+1}]$ and $v \in [v_{d_2}..v_{m_2+1}]$.

3. The curve interpolation problem

Mathematically, the curve interpolation problem may be formulated as follows: our target B-spline curve $C(t)$, of control points $(p_i)_i$ as defined in Eqn. (1), with a pre-image curve $\Phi(t) = \ < u(t), v(t) >$ in the uv parameter domain.

Accordingly, the curve on surface $S(u(t), v(t))$ would be equal to the following expression:

$$\sum_{i=0}\sum_{j=0} N_i(u(t))N_j(v(t))p_{ij} \tag{7}$$

The goal is to modify the surface $S(u, v)$ locally, so as to establish the following identity:

$$S(u(t), v(t)) = C(t) \tag{8}$$

Our interest in curves on surfaces here is just to provide a point of focus as to which parts of the surface would need to be modified in order to establish the identity describe by Eqn. (8).

We will assume here, without loss of generality, that the curve $C(t)$ is a cubic B-spline curve defined over a knot vector $\mathbf{T}$. Furthermore, we will also assume that the surface $S(u, v)$ is a bi-cubic B-spline surface defined over the knot vectors $\mathbf{U} \times \mathbf{V}$. Furthermore, as an alternative, the knot vectors could also involve an affine map.

4. The iso-parametric case

In order to motivate our discussion, we will first address the case of iso-parametric curves interpolation. In this case, interpolation is straightforward. For example, when the uv curve $\Phi(t)$ is a constant v-curve, i.e. $u(t) = t$ and $v(t) = c$, the surface curve $S(u(t), v(t))$ has the same u-basis as that of the surface $S(u, v)$; i.e. $\mathbf{U} = \mathbf{T}$, or an affine map of it.

4.1. Factorization

For any given particular parameter u, $N_i^{d_1}(u)$ may be factored out of the inner summation of Eqn. (6), since it is constant along the v direction. Thus, the global expression of Eqn. (6) may be rewritten as:

$$\sum_{i=0}^{m_1} N_i^{d_1}(u) \sum_{j=0}^{m_2} N_j^{d_2}(v)p_{ij} \tag{9}$$

By comparison with Eqn. (6), Eqn. (9) also represents a curve:

$$C'(u) \equiv \sum_{i=0}^{m_1} N_i^{d_1}(u)p'_i \tag{10}$$

where

$$p'_i \equiv \sum_{j=0}^{m_2} N_j^{d_2}(v)p_{ij} \tag{11}$$

Moreover, the curve $C'(u)$ obtained as such is obviously interpolated by the surface $S(u, v)$, something that is, again, made possible by the fact that the curve $C'(u)$ is iso-parametric with respect to parameter u of the surface.

4.2. Interpolating an iso-parametric curve

In general, when parameter v is arbitrarily selected, the summation of Eqn. (11) will then depend on $d_2 + 1$ terms (here, $v_k \le v < v_{k+1}$);

$$p'_i = \sum_{j=k-d_2}^{k} N_j^{d_2}(v)p_{ij} \tag{12}$$

In the cubic case (i.e., when $d_2 = 3$) for example, the reasoning of section 2.2 may be applied here to deduce that the summation of Eqn. (11) depends on 4 rows of the B-spline control point grid (see Fig. 4). The marked rows of the grid form what is called a B-spline *polygonal complex* [1, 11]. This may also be expressed by a $4 \times m_1$ matrix M of points.

Accordingly, the control polygon (P) represented by the following matrix multiplication:

$$\begin{bmatrix} N_k^3(v) & N_{k-1}^3(v) & N_{k-2}^3(v) & N_{k-3}^3(v) \end{bmatrix} \times M \tag{13}$$

corresponds to a B-spline curve, interpolated by the surface $S(u, v)$. Conversely, if row number k-2 of M is

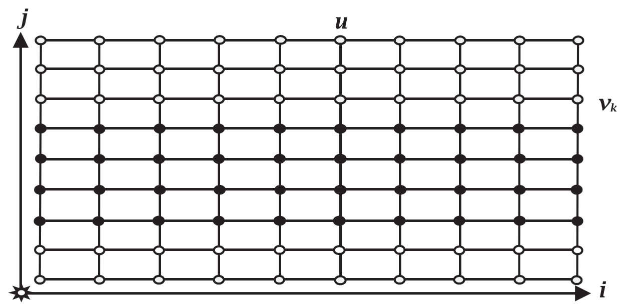

Figure 4. Polygonal Complex (in bold) corresponding to an iso-parametric curve.

replaced by the following polygon:

$$\frac{1}{N^3_{k-2}(v)}\left[-N^3_k(v) \quad -N^3_{k-1}(v) \quad 1 \quad -N^3_{k-3}(v)\right] \times M \tag{14}$$

then the resulting surface will interpolate the curve corresponding to this row.

4.3. Interpolating a curve corresponding to v knot line

However, as noted in [1], in the case of a v knot line (i.e., $v = v_k$), the summation of Eqn. (11) will then depend on only d_2 terms;

$$p'_i = \sum_{j=k-d_2}^{k-1} N_j^{d_2}(v_k)p_{ij} \tag{15}$$

In the cubic case (i.e., when $d_2 = 3$) for example, the reasoning of section 2.2 may be applied to deduce that the summation of Eqn. (15) depends on only 3 rows of the B-spline control point grid (see Fig. 5).

The marked rows form what is called a B-spline *polygonal complex* [1, 11], which may also be expressed by a $3{\times}m_1$ matrix M of points. Accordingly, the control polygon (P) represented by the following matrix multiplication:

$$\left[N^3_{k-1}(v_k) \quad N^3_{k-2}(v_k) \quad N^3_{k-3}(v_k)\right] \times M \tag{16}$$

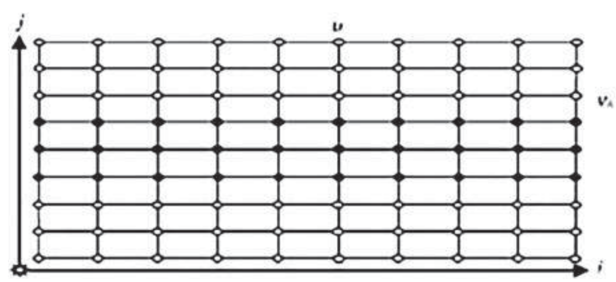

Figure 5. Polygonal Complex (in bold) corresponding to a knot line curve.

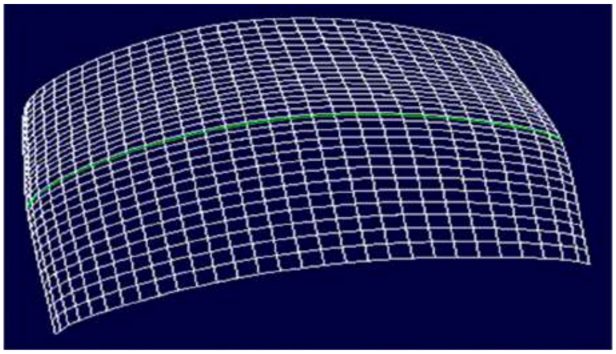

Figure 6. Curve interpolated by the surface corresponding to knot line v_k.

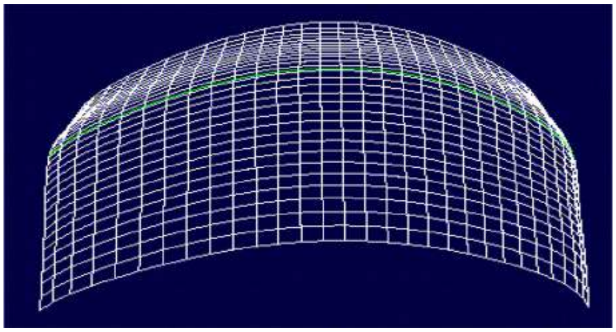

Figure 7. Surface altered to interpolate the curve of that row of the control mesh.

corresponds to a B-spline curve interpolated by the surface $S(u, v)$ (see Fig. 6).

Conversely, if the row of the grid corresponding to that row of M is replaced by the following polygon:

$$\frac{1}{N^3_{k-2}(v_k)}\left[-N^3_{k-1}(v_k) \quad 1 \quad -N^3_{k-3}(v_k)\right] \times M \tag{17}$$

then the resulting surface will interpolate the curve corresponding to that row (see Fig. 7).

5. The non-iso-parametric case

In the non-iso-parametric case, the uv curve is constant neither along the u direction nor along the v direction. In other words, the knot vector **T** of the curve is neither equal to the knot vector **U** nor the knot vector **V** of the surface. This can come under a variety of forms, also possibly involving affine maps; a simple one is depicted in Fig. 8(a).

Here, the pre-image $\Phi(t)$ in the uv parameter domain of the curve $C(t)$ would be non-iso-parametric with respect to the given tensor-product B-spline surface $S(u, v)$ (see Fig. 8(a)).

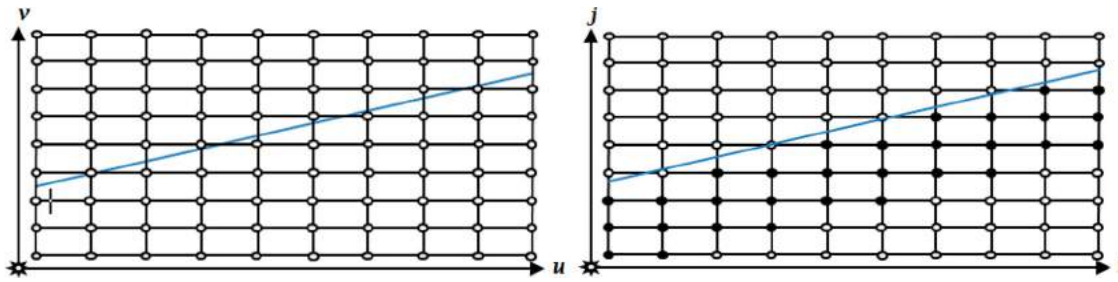

Figure 8. Surface and curve: (a) The pre-image of a non-iso-parametric curve in the *uv* domain, (b) Control point support of the surface for the curve.

The goal now is to modify the surface $S(u, v)$, so as to establish the identity of Eqn. (8). As we mentioned above, the modification of the surface will be local. This is due to the locality of the B-spline basis functions.

In fact, we only need to modify those control points for which the support of their corresponding B-spline basis intersects with the *uv* curve Φ(t). For example, if the surface is bi-cubic and all interior knots are simple (multiplicity equal to one), then all interior bases $N_i^3(u)N_j^3(v)$ have the support of a 4×4 rectangle of control points (see Fig. 8(b)) for this base. We shall mark the lower-left corner as corresponding to control point p_{ij} for this base.

For the given *uv* curve Φ(t), Fig. 8(b) shows the control points which are involved in the construction of the curve on surface *S(u(t), v(t))*. On this basis, it would seem reasonable to assume that these are the only control points of the surface that need to be repositioned in order to achieve interpolation.

It is worth noting here that the area of the surface specified by these control points may be considered as a generalization of the B-spline polygonal complex as depicted in Fig. 4, for example. However, the absence of a constant parameter here means that factorization is not possible here. Thus, in order to achieve interpolation, we need to seek an alternative approach.

In fact, the obvious option here is to think of these control points as variables then use them is a system of linear equations, whose solution would satisfy Eqn. (8). This would alter the surface to a situation that would interpolate the given non-iso-parametric curve.

5.1. B-spline multiplication and composition

The goal is to alter the surface so as to establish the identity expressed by Eqn. (8). Thus, the first step would be to make sure that the two curves $C(t)$ and $S(u(t), v(t))$ have the same degree. In fact, if the initial curve $C(t)$ is cubic and the initial surface $S(u, v)$ is bi-cubic, and if the curves $u(t)$ and $v(t)$ are both linear, then the curve-on-surface $S(u(t), v(t))$ will be of degree 6. As a result, the degree of the curve $C(t)$ should be elevated from 3 to 6 (see [14]), which will give us the curve $C'(t)$ of degree 6 with knot vector **T'** and control point sequence $(p'_k)_k$.

Next, with reference to Eqn. (7), and for each pair of indices $<i, j>$, there will be a sequence of coefficients $(\delta_{ijk})_k$ along the knot vector **T'** such that:

$$N_i(u((t))N_j(v(t)) = \sum_k \delta_{ijk} N_k(t) \qquad (18)$$

This is a classical problem of B-spline multiplication and composition. More details about the solution followed here may be found in Hu&Sun [6]. The full analysis is in E. T. Y. Lee [7] who gives a simple and quick blossoming-based algorithm to compute the B-spline coefficients from the power polynomial form of a B-spline. More literature on the subject may also be sought in Lyche and Morken [8], Morken [10] and Ramshaw [15].

5.2. The system of linear equations

Now, we can rewrite the curve-on-surface expressions in Eqn. (7) and Eqn. (8) as:

$$S(u(t), v(t)) \equiv \sum_i \sum_j N_i(u(t))N_j(v(t))p_{ij} \qquad (19)$$

which is again: $S(u(t), v(t)) \equiv \sum_i \sum_j \sum_k \delta_{ijk} N_k(t) p_{ij}$ and again:

$$S(u(t), v(t)) \equiv \sum_k \sum_i \sum_j \delta_{ijk} N_k(t) p_{ij}$$

Now, if we match that against the degree-elevated B-spline curve *C'(t)*, we obtain:

$$\sum_i \sum_j \delta_{ijk} p_{ij} = p_k{}' \qquad (20)$$

for all control points of the curve *C'(t)*.

5.3. The solution

By solving this linear system, we can obtain the new positions of the surface control points (p_{ij}) in the support of the curve depicted in Fig. 8(b). This will guarantee that the curve will be interpolated by the modified version of the surface.

However, we should perhaps note that the system of equations in Eqn. (20) might not be always solvable. In fact, the general condition for the solvability of Eqn. (20) is very complicated. For example, we cannot move a lower degree surface to a higher degree target curve in general or, in other words, we cannot move a surface with lower complexity to a target curve with higher complexity. For that, we may need to use Degree Elevation and Knots Insertion to increase the complexity of the surface.

For this reason, we seek here the use of algorithms such as the SVD (Singular Value Decomposition, cf. Numerical Recipes in C) to solve Eqn. (20), see [3] and [5]. This algorithm can find the exact solution if there is one and the least square approximation if there is no exact one. Moreover, if there happens to be more than one solution, this algorithm can find one with minimum change from original control points.

6. Conclusions and further work

This paper shows that the interpolation of iso-parametric curves could be reached rather directly, where it is sufficient to manipulate B-spline basis functions at surface level only. However, this research direction proves difficult to maintain when interpolating non-iso-parametric curves.

For this reason, the approach proposed in this paper is based on the representation of B-splines in polar form, or Blossom, as proposed in matching trimmed surfaces. This approach is useful in breaking down composition and product of B-spline basis functions, occurring in the expression of a curve on surface.

Furthermore, due to the fact that the resulting system of linear equations might not always be solvable, the suggestion of the use of the SVD algorithm is necessary to obtain an approximate solution when an exact solution is not possible to reach.

Finally, although the discussion is limited to curves which are linearly mapped in the corresponding parameter domain, the interpolation of other types of curves could be inspired. The solution proposed here is of a general nature, where the interpolation of iso-parametric curves becomes a simple particular case. The interpolation of curves with non-linear $u(t)$ and $v(t)$ could be inspired as future work.

Further work could also investigate the supply of numerical values to show how the suggested approach would work in practice and also, maybe, to get enough insights to propose conditions for the solvability of the problem in general.

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