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Time optimal driving on curvilinear path with kinematic constraints

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ABSTRACT

A time optimal driving algorithm is obtained for the moving on a curvilinear path in space. The algorithm takes velocity, acceleration and jerk as constraints. By imposing a jerk constraint, the acceleration time-derivative is limited and smooth driving is guaranteed. It is concluded that the moving object's dynamics must be analyzed directly by using curvature and torsion of the path. It is also found that the given path must possess G^2 or higher continuity for applying a jerk constraint. For a given set of velocity, acceleration and jerk constraints, it is proved that the minimum driving time depends on path length, curvature, torsion and curvature's path length derivative along the path. The resultant driving pattern guarantees minimum-time smooth driving.

KEYWORDS

Time optimal; driving pattern; curvilinear path; jerk constraint; curvature; torsion; G^2 continuity

1. Introduction

In work such as parts transport and processing, the parts or the effectors of a robot are often driven on a specified curvilinear path. Also, a car or a train often runs on a curved road or railway. When moving an object on a curved path, the force acting on the object includes not only the component in tangential direction but also the component in normal direction. Since the force acting on moving object is influenced by curvature and torsion of the path, they must be taken into account. How is an object driven through a curvilinear path as quickly as possible while limiting the force acting on it? It is the subject to be studied in this paper.

In this study, by formulating the velocity, acceleration, and jerk of a path moving object with the path curvature and torsion, the solution's uniqueness and existence of the minimum time driving pattern subject to the given constraints are clarified. Then an algorithm for calculating the minimum time driving pattern and several calculating results are provided. Also, comparisons with traditional driving method are shown and verification results with a real robot are described.

So far, cam curves have been often used for path driving [13]. These curves are formulated by some elementary functions, so they are easy to be used. However, these curves were originally created for linear motion, not for curvilinear path. Its driving velocity does not take into account the change in curvature of the path and it takes more driving time than necessary.

As a study of the shortest time driving on the curvilinear path, Bobrow et al. suggested a method by considering limitations of angular velocity and torque of actuators [1]. They expressed the dynamics of the driving system with curve length of the path. Then, the minimum time driving pattern is created by maximally accelerating and maximally decelerating segments. By this mean, the resultant driving pattern guarantees shortest time driving. However, the acceleration in acceleration-deceleration switching position is not continuous. This driving method is so-called bang-bang control, and is evidently not a smooth driving.

In order to overcome the disadvantages of the bang-bang control described above, some improving studies were carried out [3–4],[6–7],[10–12],[16]. In addition to limiting the angular velocity and the maximum torque of actuators, the limitation on torque's time derivative was also considered. However, in practical application, one wishes to limit the forces acted on an object moving on the path rather than its actuator's torque. These two subjects are actually not equivalent. For example, when rotating a manipulator's arm fixed on the vertical shaft of a motor with high speed, an object fixing on the arm top moves on a circular orbit. In this case, a strong centrifugal force will be acted on it (in radial direction), but this force is absorbed by the bearing of rotating shaft and is not reflected on rotational torque of the motor. A car passing through a curvilinear road at high speed without considering this centrifugal force may face dangers of falling or side-slipping.

In this research, the focus is put not on torque of the actuators but on object moving on its path with given kinematic constraints such as velocity, acceleration

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and jerk. Up to now, this approach has not been well studied.

2. Constraints and formulation

2.1. Kinematic constraints

In curvilinear motion, velocity, acceleration and jerk of the moving object are vectors. We take maximum absolute values of those vectors as the constraints. The velocity need to be limited for safety reasons. Limitation of maximum acceleration is important because it is proportional to the force acted on the moving object. By limiting maximum jerk, the changing rate of acceleration is restricted and smooth movement can be realized.

In actual operation, the constraints of velocity, acceleration and jerk are decided by the purpose of application. For example, when moving a fragile object such as a cake, both acceleration and jerk ought to be lowly limited. By setting the constraints suitably, it is possible to achieve time optimal driving while reducing the moving vibration.

2.2. Formulation of curvilinear motion

For brevity, we consider the moving object as a mass point. A curvilinear path is shown in Fig. 1. Here, s is the curve length from path start point, h represents total path length and $\boldsymbol{r}(s)$ represents position vector of the moving mass point. The definitions of velocity $\boldsymbol{v}$, acceleration $\boldsymbol{a}$ and jerk $\boldsymbol{j}$ are given in Eqn. (2.1).

$$\begin{cases} \boldsymbol{v} = \dot{\boldsymbol{r}}(s) = d\boldsymbol{r}(s)/dt \\ \boldsymbol{a} = \ddot{\boldsymbol{r}}(s) = d^2\boldsymbol{r}(s)/dt^2 \\ \boldsymbol{j} = \dddot{\boldsymbol{r}}(s) = d^3\boldsymbol{r}(s)/dt^3 \end{cases} \tag{2.1}$$

$$\begin{cases} d\boldsymbol{T}/ds = \kappa \boldsymbol{N} \\ d\boldsymbol{N}/ds = -\kappa \boldsymbol{T} + \tau \boldsymbol{B} \\ d\boldsymbol{B}/ds = -\tau \boldsymbol{N} \end{cases} \tag{2.2}$$

$$\begin{cases} \boldsymbol{v} = \dot{s}\boldsymbol{N} \\ \boldsymbol{a} = \ddot{s}\boldsymbol{T} + \dot{s}^2\kappa \boldsymbol{N} \\ \boldsymbol{j} = (\dddot{s} - \dot{s}^3\kappa^2)\boldsymbol{T} + (3\ddot{s}\dot{s}\kappa + \dot{s}^3\kappa')\boldsymbol{N} + \dot{s}^3\kappa\tau \boldsymbol{B} \end{cases} \tag{2.3}$$

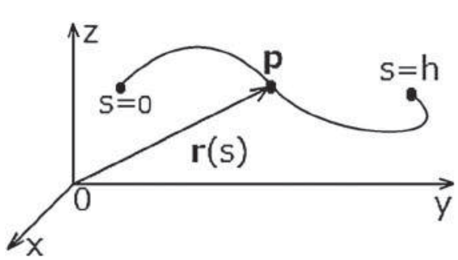

Figure 1. A particle moving on curvilinear path.

At an arbitrary point $\boldsymbol{P}$ on the curvilinear path, we express the corresponding curvature as κ and the corresponding torsion as τ. The Frenet-Serret formulas [9] are shown in Eqn. (2.2). Here, $\boldsymbol{T}$, $\boldsymbol{N}$ and $\boldsymbol{B}$ are unit vectors in tangential, normal and bi-normal directions respectively. Notice $d/dt = (ds/dt)\, d/ds$ and $dr/ds = \boldsymbol{T}$, we can combine Eqn. (2.1) and (2.2) to obtain Eqn. (2.3). Here, $\dot{s}$, $\ddot{s}$ and $\dot{s}$ are the first, second and third order derivatives of s with respect to time, κ' is derivative of the curvature to curve length. In Eqn. (2.3), $\dot{s}$ represents the velocity (tangential direction), $\ddot{s}$ and $\dot{s}^2\kappa$ represent tangential and normal acceleration component, $3\dddot{s} - \dot{s}^3\kappa^2$, $3\ddot{s}\dot{s}\kappa + \dot{s}^3\kappa'$ and $\dot{s}^3\kappa\tau$ represent the tangential, normal and bi-normal jerk components respectively.

In curvilinear motion, the velocity has only a tangential component. But acceleration has two components which are in tangential and normal direction, its component in the normal direction is proportional to curvature and square of velocity. Jerk usually has three components which are in tangential, normal and bi-normal direction. While moving on a circle with constant velocity, $\dddot{s} = 0, \ddot{s} = 0, \tau = 0, \kappa' = 0$. In this case, the jerk components in normal and bi-normal direction are zero, but its tangential component is $-\dot{s}^3\kappa^2$ because the direction of acceleration (point to the circle center) changes continuously. In order to apply jerk limitation, Eqn. (2.3) also shows that a path must possess G^2 or higher continuity if $\dot{s} \neq 0$.

2.3. Formulation of the Problem

Let V, A and J represent the constraints (the maximum absolute values of $\boldsymbol{v}$, $\boldsymbol{a}$ and $\boldsymbol{j}$), h represents the total path length, t_f represents the final moving time. Our optimizing problem can be expressed as Eqn. (2.4). The limitations of $\dot{s}$, $\ddot{s}$ and are determined in Eqn. (2.5). Here, stationary states are supposed at both start and end point.

$$\begin{cases} ds/dt = \dot{s} \\ d^2s/dt^2 = \ddot{s} \\ d^3s/dt^3 = \dddot{s} \\ 0 \le s \le h \\ 0 \le \dot{s} \le V \\ \ddot{s}^2 + \dot{s}^4\kappa^2 \le A^2 \\ \left(\dddot{s} - \dot{s}^3\kappa^2\right)^2 + \left(3\ddot{s}\dot{s}\kappa + \dot{s}^3\kappa'\right)^2 + \dot{s}^6\kappa^2\tau^2 \le J^2 \\ s(0) = 0, \quad s\left(t_f\right) = h \\ \dot{s}\,(0) = 0, \quad \dot{s}\left(t_f\right) = 0 \\ \ddot{s}\,(0) = 0, \quad \ddot{s}\left(t_f\right) = 0 \\ minimize[t_f = \int_0^h (1/\dot{s})\, ds] \end{cases} \tag{2.4}$$

$$\begin{cases} \dot{s}_H = \min\left\{V, (A/\kappa)^{1/2}\right\} \\ \ddot{s}_H = \left(A^2 - \dot{s}^4\kappa^2\right)^{1/2} \\ \ddot{s}_L = -\left(A^2 - \dot{s}^4\kappa^2\right)^{1/2} \\ \dddot{s}_H = \dot{s}^3\kappa^2 + \left(J^2 - \left(3\dot{s}\ddot{s}\kappa + \dot{s}^3\kappa'\right)^2 - \dot{s}^6\kappa^2\tau^2\right)^{1/2} \\ \dddot{s}_L = \dot{s}^3\kappa^2 - \left(J^2 - \left(3\dot{s}\ddot{s}\kappa + \dot{s}^3\kappa'\right)^2 - \dot{s}^6\kappa^2\tau^2\right)^{1/2} \end{cases} \tag{2.5}$$

Notice that the acceleration and jerk of a moving object are $\boldsymbol{a}$ and $\boldsymbol{j}$, not $\ddot{s}$ and $\dddot{s}$. The latter ones are simply used as convenient variables for finding the time optimal driving pattern. Actually, $\ddot{s}$ is the tangential component of $\boldsymbol{a}$, but $\dddot{s}$ is not even the tangential component of $\boldsymbol{j}$. The tangential component of $\boldsymbol{j}$ is $\dddot{s} - \dot{s}^3\kappa^2$ indeed.

3. Features of time optimal driving

3.1. Maximum velocity achievable

In this section, we will show that the maximum achievable velocity must be obtained in time optimal driving. We also prove that one of $\dot{s}(t)$, $\ddot{s}(\mathrm{t})$ or $\dddot{s}(t)$ must touch their limitation at any driving interval.

To make it easier to understand, we explain from a calculated pattern shown in Fig. 2. In this figure, $\dot{s}(t)$, $\ddot{s}(\mathrm{t})$, $\dddot{s}(\mathrm{t})$, and their limitations are shown. This is a calculated time optimal pattern for driving on a circle path. At time $\mathrm{t} = \mathrm{t}_0 = 0$ (start point), $\dot{s}(\mathrm{t}_0) = 0$, $\ddot{s}(\mathrm{t}_0) = 0$. At time $\mathrm{t} = \mathrm{t}_5$ (end point), $\dot{s}(t_5) = 0$, $\ddot{s}(\mathrm{t}_5) = 0$. Here, $\dot{s}_H$, $\ddot{s}_{\mathrm{H}}$, $\ddot{s}_{\mathrm{L}}$, $\dddot{s}_H$ and $\dddot{s}_L$ are calculated by Eqn. (2.5).

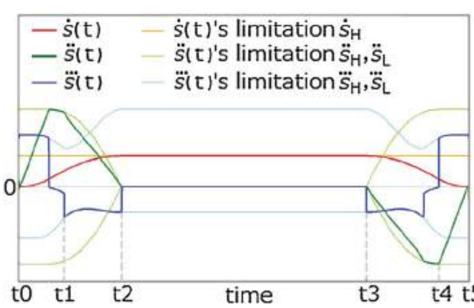

Figure 2. A time optimal driving pattern on a circle path.

$[\mathrm{t}_0, \mathrm{t}_1]$ is the accelerating interval with best effort using limitations of $\dddot{s}_H$ and $\ddot{s}_{\mathrm{H}}$. $[\mathrm{t}_1, \mathrm{t}_2]$ is the decelerating interval with best effort using limitations of $\dddot{s}_L$. At time t_2, $\ddot{s}$ become zero and $\dot{s}$ reaches its limitation $\dot{s}_H$. Notice that this driving pattern of $\dot{s}$ is the fastest one in interval $[\mathrm{t}_0, \mathrm{t}_2]$. It is because in interval $[\mathrm{t}_0, \mathrm{t}_1]$, no higher $\dot{s}$ can be obtained due to limitations of $\dddot{s}_H$ and $\ddot{s}_{\mathrm{H}}$. Also in interval $[\mathrm{t}_1, \mathrm{t}_2]$, if any higher $\dot{s}$ is assigned, the limitation $\dot{s}_H$ must be broken even with full braking. In interval $[\mathrm{t}_2, \mathrm{t}_3]$, $\dot{s} = \dot{s}_H$, this is evidently the fastest driving in $[\mathrm{t}_2, \mathrm{t}_3]$.

In interval $[\mathrm{t}_3, \mathrm{t}_5]$, it is the braking pattern using $\dddot{s}_L$, $\ddot{s}_{\mathrm{L}}$, and $\dddot{s}_H$. In interval $[\mathrm{t}_4, \mathrm{t}_5]$, $\ddot{s}$ is brought to zero by using $\dddot{s}_H$ for satisfying boundary conditions at end point. This driving pattern of $\dot{s}$ is also the fastest one in interval $[\mathrm{t}_3, \mathrm{t}_5]$. In fact, if any higher $\dot{s}$ is assigned in this interval, the boundary conditions at end point cannot be satisfied. We call this braking pattern as time optimal stopping pattern.

Based on the above analysis, this driving pattern of $\dot{s}$ is the fastest one in whole interval of $[\mathrm{t}_0, \mathrm{t}_5]$ which satisfies the given constraints. The time optimal driving pattern consists of maximum accelerating and maximum decelerating segments. This is always true because more complicated cases can be subdivided into the same segments as showing in Fig. 2.

Next, we prove that one of $\dot{s}(t)$, $\ddot{s}(\mathrm{t})$ or $\dddot{s}(t)$ must touch their limitation at any driving time. First, we assume that a time optimal pattern is obtained and none of $\dot{s}(t)$, $\ddot{s}(\mathrm{t})$ or $\dddot{s}(t)$ touches their limitation during some interval $[\mathrm{t_a}, \mathrm{t_b}]$. Let the corresponding path segment is $[\mathrm{s_a}, \mathrm{s_b}]$. For this interval, we define $\mathrm{v(s)} = \epsilon(\mathrm{s} - \mathrm{s_a})^4(\mathrm{s} - \mathrm{s_b})^4$. Here, ϵ is a positive real number with units of $\mathrm{m}^{-3}\mathrm{s}^{-1}$. Notice that $d/dt = (ds/dt)d/ds$ and differentiate v(s) with time, $\dot{v}(\mathrm{s})$ and $\ddot{\mathrm{v}}(\mathrm{s})$ can be obtained. At $\mathrm{t_a}$,$\mathrm{t_b}$, all of v(s), $\dot{v}(\mathrm{s})$ and $\ddot{\mathrm{v}}(\mathrm{s})$ are zero. In interval $(\mathrm{s_a}, \mathrm{s_b})$, all of v(s), $\dot{v}(\mathrm{s})$ and $\ddot{\mathrm{v}}(\mathrm{s})$ are positive. Then, we replace $\dot{s}$, $\ddot{s}$ and $\dddot{s}$ with $\dot{s} + \mathrm{v}$, $\ddot{s} + \dot{v}$ and $\dddot{s} + \ddot{v}$ in interval $[\mathrm{s_a}, \mathrm{s_b}]$. In doing so, we get a faster driving pattern which also satisfies all given constraints as long as ϵ is small enough. But this contradicts the fact that the original driving pattern is time optimal as we have assumed. So, for a time optimal driving pattern, one of $\dot{s}(t)$, $\ddot{s}(t)$ or $\dddot{s}(t)$ must touch their limitation at any driving interval.

3.2. Uniqueness and existence of the solution

As discussed in 3.**1**, for a time optimal driving pattern, its $\dot{s}(\mathrm{t})$ must be the maximum achievable one at any time during driving. First, we assume there are two such solutions with moving time T1, T2 and denote them as $\dot{s}_1(\mathrm{t})$, $\dot{s}_2(\mathrm{t})$ respectively. Evidently T1 = T2 because both ones are time optimal driving. Also, because both $\dot{s}_1(\mathrm{t})$ and $\dot{s}_2(\mathrm{t})$ are the maximum achievable ones during driving, $\dot{s}_1(\mathrm{t}) \geq \dot{s}_2(t)$ and $\dot{s}_2(\mathrm{t}) \geq \dot{s}_1(t)$ must hold. This means $\dot{s}_1(\mathrm{t}) = \dot{s}_2(t)$ and T1 = T2. So, time optimal driving solution is unique.

Regarding the existence of the solution, it depends on the constraints, the boundary condition and the path shape. First, the constraints V, A and J must be greater than zero. If the boundary condition is stationary, a solution must exist as long as it is driven at a velocity low enough. However, if the boundary condition is not stationary, there is a possibility that the solution does not exist. For example, when a very high velocity is assigned at starting point and a stationary state is assigned at ending point, you may not avoid breaking the constraints even if you decelerate with maximum effort. This can be confirmed only by calculating.

4. Algorithm and results

4.1. Algorithm

As shown in Eqn. (2.4), our problem is formulized as simultaneous differential equations with assigned boundary condition and nonlinear constraints on state variables. The solution to this kind of problem has not been well established [2],[8],[14–15]. The algorithm proposed here can be considered as the jerk extension of Bobrow et al. The algorithm uses bisection method [14]. Though this is not a fast algorithm, but it can give accurate results and it is reliable. First, we define **AME** and **BME** driving pattern as follows.

AME: Accelerating with Maximum Effort by using $\dddot{s}_H$ while $\ddot{s} < \ddot{s}_H$, otherwise using $\ddot{s}_H$.

BME: Braking to stop with Maximum Effort by using $\dddot{s}_L$ while $\ddot{s} > \ddot{s}_L$, otherwise using $\ddot{s}_L$. At the braking end, both$\dot{s} = 0$ and $\ddot{s} = 0$ must be satisfied. It means that $\ddot{s}$ must be brought to zero by using $\dddot{s}_H$ when reaching stop. The switch timing to use $\dddot{s}_H$ can be calculated by bisection method.

Calculating Procedure

(1) Set $s_1 = 0$, $\dot{s}_1 = 0$ and $\ddot{s}_1 = 0$.
(2) From s_1, $\dot{s}_1$ and $\ddot{s}_1$ state, calculate **AME** until it contradicts the given constraints. Denote the final state as s_2, $\dot{s}_2$ and $\ddot{s}_2$.
(3) For $[s_1, s_2]$, using bisection method to calculate the optimal point which switches from accelerating to braking. This procedure is an iterative calculating of **AME** and **BME**.
(4) If s at the end of Step.3 just reaches final position $s = h$, all calculating is finished. Otherwise, move one mini step along the calculated **BME**. Set the new state to s_1, $\dot{s}_1$ and $\ddot{s}_1$, then repeat Step.2.

The iterative calculation of **AME** and **BME** is the core of this algorithm. Notice $\ddot{s}$ must be continuous because of jerk constraint.

4.2. Calculating results

As calculating examples, the time optimal driving pattern on a s-shape path and on a helix path are shown in Fig. 3 and Fig. 4. The path shape, curvature, torsion and the curvature's derivative are plotted with respect to

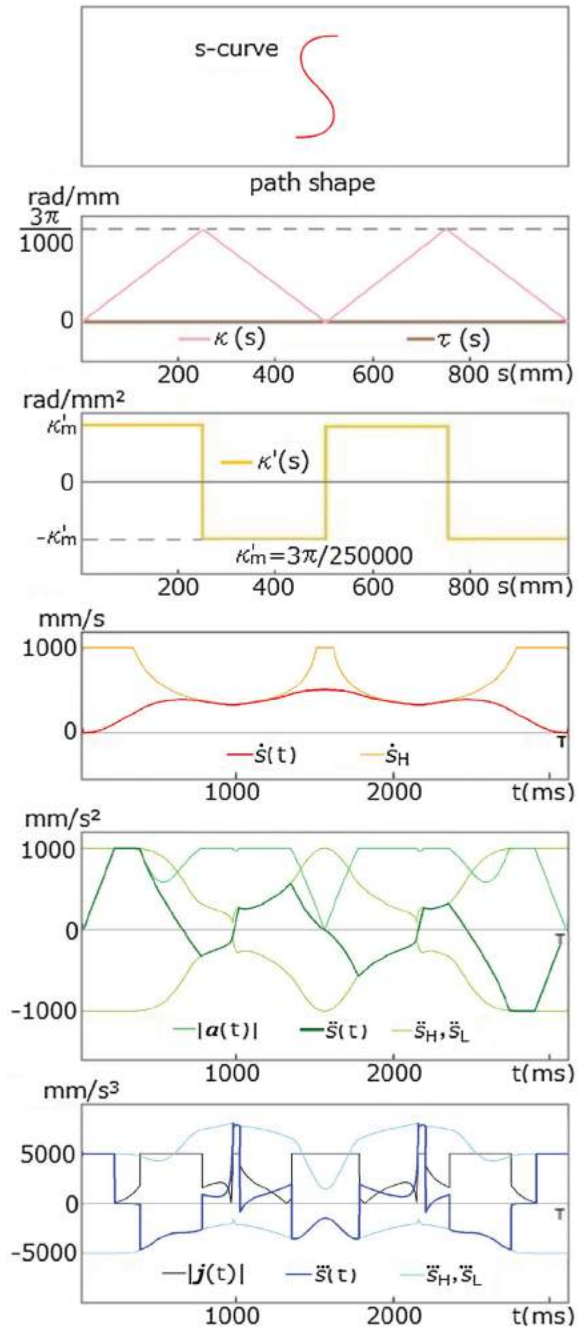

Figure 3. Time optimal driving pattern on a S-curve path of $h = 1000$ mm.

curvilinear path length. The speed, acceleration and jerk of the calculated time optimal driving pattern are plotted with respect to time. $\dot{s}(t)$, $\ddot{s}$(t) and $\dddot{s}(t)$ are drawn with thick lines and the others are drawn with thin lines. The calculation conditions are listed as follows.

- Constraints: $V = 1000$ mm/s, $A = 1000$ mm/s^2, $J = 5000$ mm/s^3.
- Path length $h = 1000$ mm.
- Mini time step adopted for calculation: Δt= 1 ms.

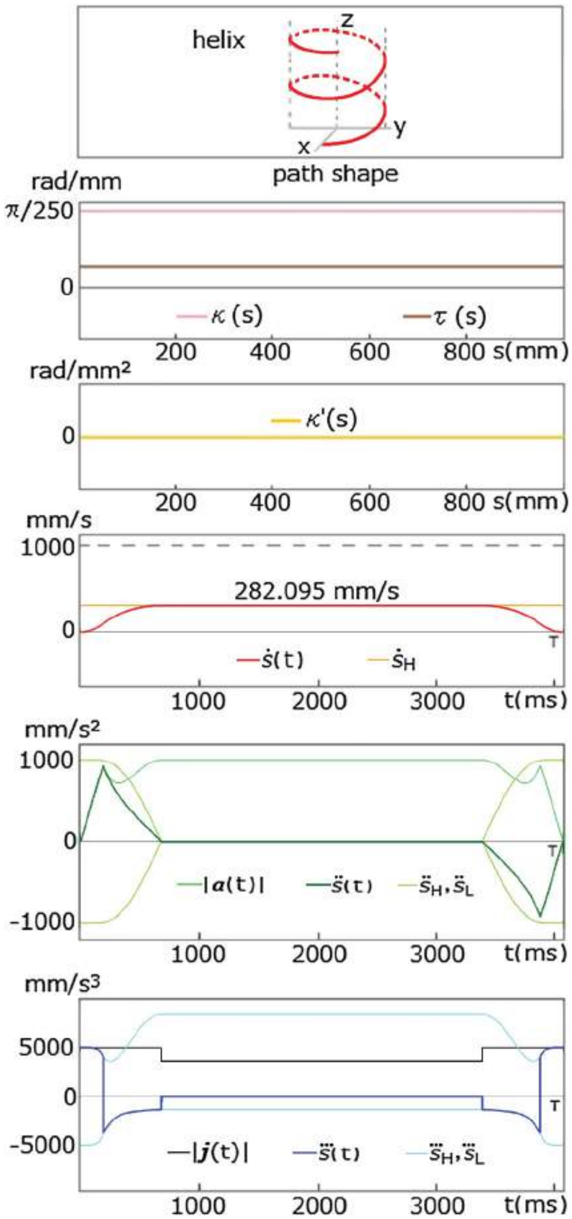

Figure 4. Time optimal driving pattern on a helix path of $h = 1000$ mm.

On the s-shape path, the curvature is zero at the starting point, the midpoint and the end point. The path has a segmental linear curvature distribution. Since the curvature changes along the path, $\dot{s}(t)$, $\ddot{s}(t)$ and $\dddot{s}(t)$ possess complicated shapes. The maximum curvature is $\kappa_{\max} = 3\pi/1000$ rad/mm. The required driving time is T = 3129 ms.

On the helix path, the curvature and torsion are constant. Its curvature is $\kappa = \pi/250$ rad/mm and its torsion is $\tau = \pi/1000$ rad/mm. The required driving time is T = 4097 ms. Notice that $\ddot{s}$(t) and $\dddot{s}(t)$ also possess complicated shapes even the curvature and torsion are constant on whole path. If torsion is assigned to zero, this helix will become two turn circles and the driving time will be T = 4096 ms, only 1ms shorter than before. It shows torsion contributes very little on driving time.

4.3. Comparison with cam driving curves

Driving time comparison between the traditional Cam driving curve and the time optimal driving pattern are shown in Tab. 1. Typically, modified constant velocity, cycloid and 5-order Cam driving curves are used. Those results are calculated for the s-curve path with the same constraints described above. Fig. 5 shows the calculated driving pattern when using modified constant velocity driving curve on the same path of Fig. 3. Here, its dimensionless parameters are T1 = 0.125, T2 = 0.25, T3 = 0.75, T4 = 0.875. Because Cam driving curves cannot adjust its velocity according to the path curvature, it always takes longer driving time than that of time optimal driving.

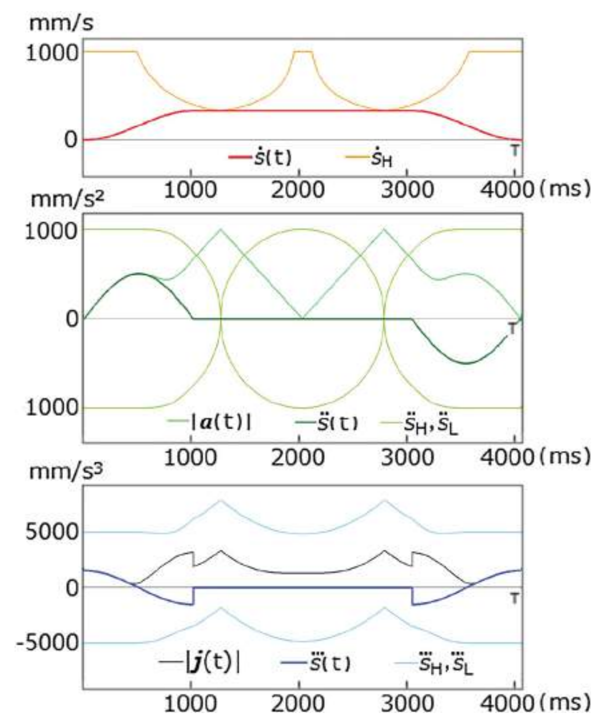

Figure 5. Modified constant velocity driving patterns on the s-curve of Fig. 3.

Table 1. Driving time comparison for the s-curve path of Fig. 3.

Time optimal	Modified constant velocity	Cycloid	5-order
3129 ms	4093 ms	5177 ms	4914 ms

4.4. Verification using real manipulator

The actual verification is shown in Fig. 6 and Fig. 7. This is an experiment of tracking s-shape path using a

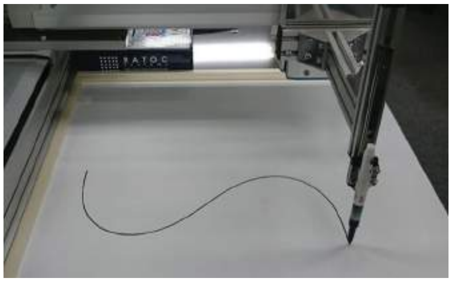

Figure 6. Plot s-curve by a manipulator.

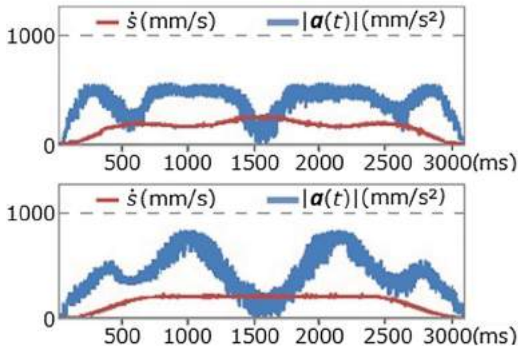

Figure 7. Driving results of Fig. 6 by using time optimal pattern (top graph) and using modified constant velocity Cam pattern (bottom graph).

manipulator with orthogonally driving arms. The driving data to motors are sending via EtherCAT which transmits data at high speed with 1 ms cycle [5]. Since the movable range of this manipulator cannot draw s-shape of 1000 mm length, a half scale condition is adopted as follows.

- Constraints: $V = 500\,\text{mm/s}$, $A = 500\,\text{mm/s}^2$, $J = 2500\,\text{mm/s}^3$.
- Path length $h = 500\,\text{mm}$.
- Mini time step adopted for calculation: $\Delta t = 1\,\text{ms}$.

Fig. 7 shows the measured result (created with the angle sensor on motor shaft) by using both time optimal driving pattern and traditional cam driving pattern (the modified constant velocity Cam curve). To obtain the same driving time, the maximum acceleration when using the time optimal pattern is lower than that of using the modified constant velocity Cam pattern. This means smoother curvilinear movement is realized.

5. Consideration on actuator's torque

Up to now, we focused on time optimal moving on a curvilinear path with given kinematic constraints such as velocity, net acceleration and net jerk. We consciously take no consideration on structure, dynamics and control of any actuator. By doing so, we obtain some general results which depend only on path shape (i.e. its curvature and torsion). These results can be widely applied to robot effectors, cars or trains moving on a curvilinear path as long as the driving system possesses enough abilities and no actuator runs into saturation.

However, in real application, some actuators may be run into saturation. This will happen if acceleration is assigned beyond its actuator's ability. In this case, much more constraints (such as limitation on actuator's torque) must be added to our original problem. In fact, the actuator's dynamics had been studied by many other ones [1–4], [6–7], [11]. In general, the dynamics of a manipulator with n actuators can be expressed as Eqn. (5.1). Here, $\boldsymbol{\tau}$ is the $n \times 1$ torque vector and $\boldsymbol{\theta}$ is the $n \times 1$ joint angle vector of the actuators, $\dot{\boldsymbol{\theta}}$ and $\ddot{\boldsymbol{\theta}}$ are the corresponding angular velocity and angular acceleration vector respectively, $\mathrm{M}(\boldsymbol{\theta})$ is the $n \times n$ mass matrix of the manipulator, $\mathrm{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ is the $n \times 1$ vector of friction and centrifugal terms and $\mathrm{G}(\boldsymbol{\theta})$ is the $n \times 1$ vector of gravity terms. Notice that $\mathrm{M}(\boldsymbol{\theta})$, $\mathrm{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ and $\mathrm{G}(\boldsymbol{\theta})$ depend on part mass, structure and posture of the manipulator. Because $\boldsymbol{\theta}$ is determined by path positions, this can be expressed as $\boldsymbol{\theta} = \boldsymbol{\theta}(s)$. Using s, Eqn. (5.1) can be written as Eqn. (5.2). Here, $\mathbf{c}(s, \dot{s})$ and $\boldsymbol{b}(s, \dot{s})$ are $n \times 1$ vectors depending on s and $\dot{s}$. Let $\boldsymbol{\tau}_h$ and $\boldsymbol{\tau}_l$ express the $n \times 1$ maximum and minimum torque vectors of the actuators, the torque constraints can be written as Eqn. (5.3). For example, if we have 3 actuators in a driving system, its torque constraints can be expressed as Eqn. (5.4). From Eqn. (5.4), we conclude Eqn. (5.5).

$$\boldsymbol{\tau} = \mathrm{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathrm{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathrm{G}(\boldsymbol{\theta}) \tag{5.1}$$

$$\boldsymbol{\tau} = \mathbf{c}(s, \dot{s})\ddot{s} + \mathbf{b}(s, \dot{s}) \tag{5.2}$$

$$\boldsymbol{\tau}_l \le \mathbf{c}(s, \dot{s})\ddot{s} + \mathbf{b}(s, \dot{s}) \le \boldsymbol{\tau}_h \tag{5.3}$$

$$\begin{cases} \tau_{1l} \le \mathrm{c}_1(s, \dot{s})\ddot{s} + \mathrm{b}_1(s, \dot{s}) \le \tau_{1h} \\ \tau_{2l} \le \mathrm{c}_2(s, \dot{s})\ddot{s} + \mathrm{b}_2(s, \dot{s}) \le \tau_{2h} \\ \tau_{3l} \le \mathrm{c}_3(s, \dot{s})\ddot{s} + \mathrm{b}_3(s, \dot{s}) \le \tau_{3h} \end{cases} \tag{5.4}$$

$$\begin{cases} f_1(s, \dot{s}) \le \ddot{s} \le g_1(s, \dot{s}) \\ f_2(s, \dot{s}) \le \ddot{s} \le g_2(s, \dot{s}) \\ f_3(s, \dot{s}) \le \ddot{s} \le g_3(s, \dot{s}) \end{cases} \tag{5.5}$$

$$\begin{cases} \ddot{s}_{max} = \min\{\ddot{s}_H, g_1, g_2, g_3\} \\ \ddot{s}_{min} = \max\{\ddot{s}_L, f_1, f_2, f_3\} \end{cases} \tag{5.6}$$

To include torque constraints of actuators, we need to replace $\ddot{s}_H$ and $\ddot{s}_L$with $\ddot{s}_{max}$ and $\ddot{s}_{min}$ expressed in Eqn.

(5.6). This only inserts more constraints to our original problem and algorithm 4.1 keeps the same. Furthermore, to achieve a designed driving pattern precisely, the control system's response is also important. In fact, when the control system's response is too slow, some driving pattern cannot be achieved accurately, while the control system's response is too quick, the moving will become noisy. It is worthwhile to make the control system's response just suitable for the designed driving pattern. This can be done by simulations or practical experiments.

6. Conclusions

In this paper, an algorithm for calculating time optimal driving pattern on a curvilinear path is obtained. The driving pattern satisfies constraints of velocity, acceleration and jerk. By applying restraint of jerk, noise, vibration and mechanical fatigue can be reduced and smooth driving is obtained. Our analysis also leads to following conclusions:

- The dynamics of a moving object on a path has to be calculated directly by path curvature and torsion rather than by torques of actuators.
- Compare with cam curves, when applying the same constraints, time optimal pattern provide shorter driving time. When assigning same driving time, time optimal pattern provides lower peek acceleration.
- In time optimal driving pattern, one of $\dot{s}(t)$, $\ddot{s}(t)$ or $\dddot{s}(t)$ must touch their limitation at any driving interval.
- Although the influence of path curvature on driving time is large, the influence of path torsion on driving time is little.
- If the boundary condition is stationary, the solution exists uniquely. If the boundary condition is not stationary, there may be no solution for some initial states and constraints.

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