A Heuristic Knot Placement Algorithm for B-Spline Curve Approximation

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ABSTRACT

A new knot placement algorithm for B-spline curve approximating to dense and noisy data points is presented in this paper. In this algorithm, the discrete curvature of the points, in contrast to the points themselves as in the traditional approaches, is smoothed to expose the curvature characteristics of the underlying curve of the data. With respect to the smoothed curvature, knots are placed to satisfy a heuristic rule. Experimental results are included to demonstrate the validity of this algorithm.

Keywords: knot placement, curve approximation, filtering, locally small deflection

1. INTRODUCTION

Reverse engineering (RE) starts with a physical model and reconstructs its geometric model from coordinate data acquired with a measuring system in order to create and/or refine the digital model. Advances in range image acquisition and other three-dimensional digitizing devices allow us to acquire very dense data points from physical objects. Although these devices are of high fidelity, measurement errors are still unavoidable due to the surface attributes of the physical object, the impact of the environment and the uncertainty of the measuring system. Consequently, the measured data points are often noisy as well as dense.

For the data points that are organized in the form of scan lines, a two step process, curve fitting and then surface lofting, is usually employed to reconstruct surfaces from the data. Generally, B-spline curve approximation instead of interpolation is preferred for dense and noisy data to create the curves. However, although curve approximation is well understood, the knot placement problem has not been dealt with satisfactorily, especially when dense and noisy data points are to be approximated. In this case, smoothing and re-sampling are usually employed to pre-process the data in present applications in order to facilitate the placement of knot and improve the performance of curve approximation [6], [7]. But the data smoothing and re-sampling operations highly depend on the intervention of the designers. The features are often blurred and the original design intent is lost if no special care is taken [8].

It has been established that the choice of knots has considerable effect on the shape of the curve [2]. An unreasonable knot vector may introduce unpredictable and unacceptable shape. In curve interpolation, the placement of knots is straightforward. On the contrary, in curve approximation, it is difficult to determine the amount and distribution of knots. Generally, a curve error bound is specified as an input together with the data points to be fitted. The amount and distribution of the knots, which are required to satisfy the bound, are both unknown in advance. Therefore, in theory, knot placement is a multivariate and multimodal nonlinear optimization problem [14].

In applications, curve approximation methods are generally iterative. Roughly speaking, these methods proceed into one of the two ways [11]: (1) start with the minimum or a small number of knots and iteratively increase the amount of knots to satisfy the error bound; (2) start with the maximum or many knots and iteratively reduce the amount of knots to satisfy the error bound. When the amount of points is very large, these methods become time-consuming if the initial knots are not well determined. Unfortunately, the problem of determining the initial knots is not well addressed in the literatures.

An exception to the above ways is the algorithm proposed by Razdan [12]. This algorithm is restricted to smooth data points. Similar algorithm was employed by Hölzle [5] to approximate a polygon to a curve. When the number of knots is given in advance and the points are reasonably distributed with regard to the curvature of the underlying curve, average knot method [9], [10] can be used to determine the distribution of the knots. However, this method highly relies on the data pre-processing operations such as smoothing and re-sampling. Furthermore, if an error bound is to be satisfied, this method becomes a trial-and-error method.

In this paper, a new knot placement method for B-spline curve approximation to dense and noisy data points is discussed. In order to preserve the shape features obliterated by the noise and reduce the time consumption, the discrete curvature of the points, in contrast to the points themselves, is smoothed, and knots are automatically placed with respect to the smoothed curvature to satisfy a heuristic rule presented in this paper without iterative calculations of the approximating curves.

The organization of this paper is as follows. A brief introduction of B-spline curve approximation is given in Section 2. In Section 3, the discrete curvature of the points is stated, followed by Section 4, where the digital filter is described. The knot placement algorithm is presented in Section 5. Examples are shown in Section 6 to demonstrate the effectivity of the presented algorithm, followed by a conclusion section that closes the paper.

2. B-SPLINE CURVE APPROXIMATION

A k^{th}-order B-spline curve is defined by

$$\mathbf{C}(u) = \sum_{i=0}^{n} N_{i,k}(u)\mathbf{P}_i ,\ u \in \left[t_{k-1}, t_{n+1}\right] \tag{1}$$

where $\{\mathbf{P}_i\}$ are the control points, and $\{N_{i,k}(u)\}$ are the k^{th}-order B-spline basis functions defined on the knot vector $\mathbf{T} = \{t_j\},\ j = 0,\ldots,n+k$. In this paper, attention is concentrated on planar end-point interpolating cubic curves with simple internal knots, i.e. the multiplicities of all internal knots are restricted to 1.

Given data points $\{\mathbf{d}_i\}$, and associating parameters $\{u_i\}$, $i = 0,\ldots,m$. The approximating curve $\mathbf{C}(u)$ in the least square sense is defined by [2]

$$\text{minimize} \sum_{\text{i}=0}^{m} \left\|\mathbf{d}_i - \mathbf{C}(u_i)\right\|^2 \tag{2}$$

For B-spline curves, the normal equation is

$$\mathbf{MP} = \mathbf{Q} \tag{3}$$

where

$\mathbf{M} = [\sum_{i=0}^{m} N_{l,k}(u_i)N_{j,k}(u_i)]$,

$\mathbf{P} = [\mathbf{p}_j]$ and

$\mathbf{Q} = [\sum_{i=0}^{m} \mathbf{d}_i N_{l,k}(u_i)]$, $l = 0,\ldots,n, j = 0,\ldots,n.$

Constraints can be incorporated into Eqn. 2 using Lagrange multipliers [11].

The matrix $\mathbf{M}$ is singular if and only if there is a span $[t_j, t_{j+k}]$, $j = 0,\ldots,n$ that contains no u_i. This fact is known as Schoenberg-Whitney condition for unconstrained curve fitting [2]. Therefore, if there is no span $[t_j, t_{j+k}]$, $j = 0,\ldots,n$ that contains no u_i, i.e. $\mathbf{M}$ is not singular, the control points can be obtained by solving Eqn. 3.

Parameterization of the points is well discussed and many methods have been proposed [2]. In this paper, chord length parametrization method is employed. The remaining problem, which is to determine a reasonable knot vector, will be discussed in the following sections.

3. DISCRETE CURVATURE OF DATA POINTS

Generally, the scan lines used in curve approximation are ordered and distributed in two-dimension, or they can be sorted to get an ordered point set according to certain criteria.

For an ordered point set $\{\mathbf{p}_i, (i = 0,\ldots,n)\}$, the discrete curvature k_i at point $\mathbf{p}_i$ $(i = 1,\ldots,n-1)$ can be defined as the inverse of the radius r_i of the circle passing through the three points $\mathbf{p}_{i-1}$, $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$, as illustrated in Fig. 1.

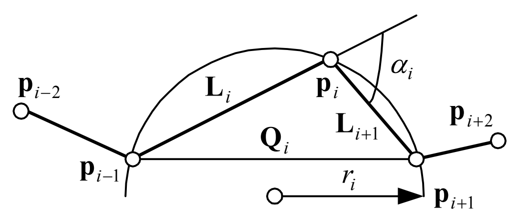


Fig. 1. Discrete curvature of ordered points

The signed discrete curvature can be expressed as [8]

$$k_i = sign(\Delta \mathbf{p}_{i-1}\mathbf{p}_i\mathbf{p}_{i+1}) \frac{2\sin(\alpha_i)}{Q_i} \tag{4}$$

where

$\mathbf{Q}_i = \mathbf{p}_{i+1} - \mathbf{p}_{i-1}$,

$Q_i = \left\|\mathbf{Q}_i\right\|$,

$\mathbf{L}_i = \mathbf{p}_i - \mathbf{p}_{i-1}$,

$L_i = \left\|\mathbf{L}_i\right\|$,

$\Delta \mathbf{p}_{i-1}\mathbf{p}_i\mathbf{p}_{i+1} = \det(\mathbf{L}_i, \mathbf{L}_{i+1})$,

$\cos(\alpha_i) = \dfrac{\mathbf{L}_i \cdot \mathbf{L}_{i+1}}{L_i L_{i+1}}$.

Hamann and Chen [3] presented a more complex scheme to estimate the discrete curvature by computing a locally interpolating quadratic polynomial. However, due to the noise in the measured data points, the

discrete curvature changes very frequently. Consequently, this quadratic polynomial scheme has no advantage over the circle scheme in respect of exposing the curvature characteristic of the underlying curve.
The discrete curvature of a noisy point set is also flawed by noise. As a matter of fact, noise in the curvature is more severe than the data themselves. In this paper, the discrete curvature is considered as equally spaced digital signal and processed using digital signal processing methods to find out the tendency and characteristics of the curvature of the underlying curve.

4. SMOOTHING OF DISCRETE CURVATURE

Like many other situations, when the discrete curvature is considered as digital signal, noise all but obliterates the signal of interest. So a lowpass filter is employed to smooth the discrete curvature.
Filters can be grouped into two categories [4]: Finite Impulse Response (FIR) filters and Infinite Impulse Response (IIR) filters. Comparing their performance, a FIR filter is used in this paper. Suppose that the sequence of numbers $\{v_n\}$ is such a set of equally spaced measurements of some quantity $v(t)$, where n is an integer and t is a continuous variable. Typically, t represents time, and $v_n = v(n)$. FIR filters are defined by [4]

$$y_n = \sum_{k=-\infty}^{\infty} c_k v_{n-k} \tag{5}$$

The coefficients c_k are the constants of the filter, the v_{n-k} are the input data, and the y_n are the outputs. In practice, the number of products must be finite, and the length of the run of nonzero coefficients c_k is shorter than the run of data y_n. A lowpass filter means that the low frequencies pass through and the high frequencies are stopped, with a transition zone between the passband and stopband of frequencies. Hence, Lowpass filters are employed to smooth out the high frequency noise in the signal.
When the signal is masked by a large amount of noise, any small peaks left in the spectrum of the signal after filtering out the noise might be either from the original signal or from ripples in the transfer function used in the filtering process. Although careful analysis could separate the two, the problem can also be avoided if a class of filters that vary smoothly rather than ripple is used. By "vary smoothly", we mean that the filters are monotone over long intervals of the frequency band.
Due to the nature of knot placement, the resulting sequence of the filtering should have precisely zero-phase distortion. In this paper, this is implemented by processing the input data in both forward and reverse directions. After filtered in the forward direction, the sequence is reversed and run back through the filter.

5. KNOT PLACEMENT

The knot vector plays an effective role in retrieving the underlying curve of the data points. In particular, given data points with considerable curvature variance of the underlying curve, the reconstructed curve may be significantly different from the underlying curve if the knots are not distributed reasonably in accordance with the varying of the curvature.
In this section, we will give a heuristic rule for knot placement, and subsequently, the knot placement algorithm for B-spline curve approximation to dense and noisy data.

5.1 A Heuristic Rule for Knot Placement

Su and Liu [13] demonstrated that, given points $\mathbf{p}_i$ $(i = 0,\ldots,n)$ (see Fig. 2), the interpolating spline curve is of locally small deflection if $\alpha_i \le \pi/6$, which means cubic spline curve could be used to interpolate the given points. Here, α_i is the angle between vectors $\mathbf{L}_{i+1}$ and $\mathbf{L}_i$, as shown in Fig. 2, with $\mathbf{L}_i = \mathbf{p}_i - \mathbf{p}_{i-1}$ $(i = 1,\ldots,n)$.

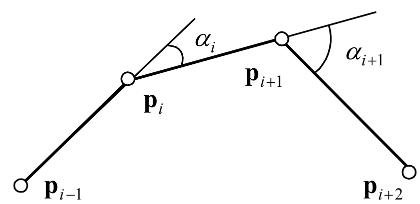


Fig. 2. Point distribution of small deflection spline

If the given points satisfy the above condition, the interpolating curve will approximate the underlying curve. Then a heuristic rule for knot placement in curve approximation is deduced from the above condition, i.e. if the points corresponding to the knots can satisfy this condition, the reconstructed curve will be a good approximation to the given data points. Hence, the problem of knot placement becomes how to select points, where knots are placed, from the given data points to fulfill the above rule.

5.2 The Algorithm

Unlike the problem of piecewise linear curve approximation to freeform curves discussed in [3] and [5], the feature points, such as curvature extrema and inflection points, are critical to retrieve the underlying curves of the data points. These kinds of points play a basic role in constraining the overall shape of the reconstructed curves. They also tend to impact the quality of the reconstructed curves. So the parameters of the feature points and the endpoints are set to be knots. In this paper, three kinds of points are defined as feature points (In the following, $k(u_i)$ is abbreviated to k_i).

(1) If $(k_i - k_{i-1})(k_i - k_{i+1}) > 0$, u_i is set to be knot. In the curvature plot, this kind of feature points corresponds to the curvature extrema, such as (u_3, k_3) and (u_6, k_6) in Fig. 3.

(2) If $(k_i - k_{i-1})(k_i - k_{i+1}) = 0$ and $|k_i - k_{i-1}| + |k_i - k_{i+1}| > 0$, u_i is set to be knot. In the curvature plot, this kind of feature points corresponds to the points connecting horizontal line segments with non-horizontal curve segments, such as (u_1, k_1), (u_2, k_2) and (u_7, k_7) in Fig. 3.

(3) In the case $k_i k_{i+1} < 0$, if $|k_i| < |k_{i+1}|$, u_i is set to be knot; otherwise, u_{i+1} is set to be knot. In the curvature plot, this kind of feature points corresponds to the points nearest to the zero-crossing points, such as (u_4, k_4) in Fig. 3.

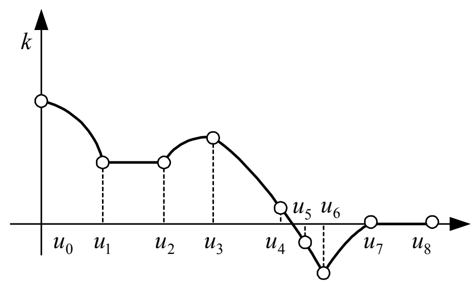
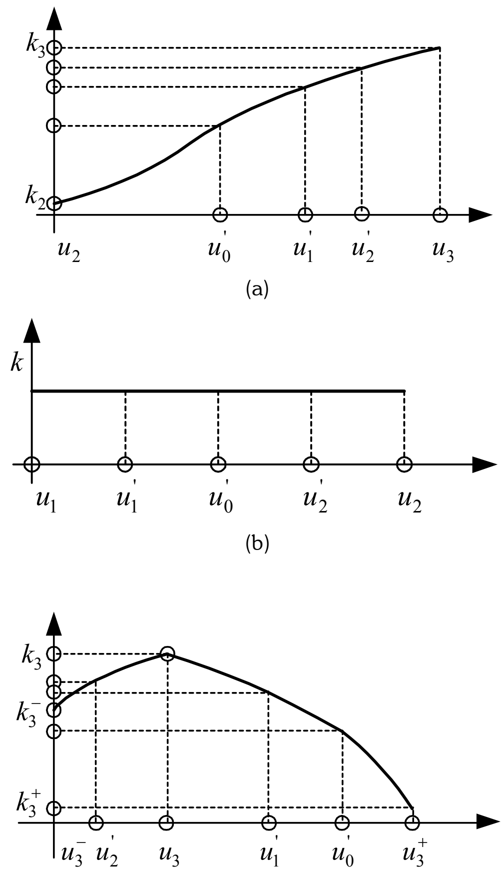


Fig. 3. Segmentation of discrete curvature

As the curvature of the points between any two adjacent feature points or the endpoints and their adjacent feature points is monotone, curvature is used as a function for knot placement in succession. Firstly, each subset is bisected iteratively with respect to their curvature and the parameters nearest to the bisecting positions are set to be knots to make all internal data points corresponding to the knots satisfy the heuristic rule of knot placement. This process is shown in Fig. 4(a). The horizontal axis indicates the parameters of the points, and the vertical axis indicates the discrete curvature of the points. $u_0^{'}$, $u_1^{'}$ and $u_2^{'}$ are the knots inserted into $[u_2, u_3]$ and the subscript variables indicate the order of the insertion of the knots. When the curvature of a subset is constant, i.e. the underlying curve of the subset is a circle or a line, the parameter axis is bisected iteratively and knots are placed at the points whose parameters are nearest to that of the bisecting positions (see Fig. 4(b)).

At last, the feature points are tested to verify if they satisfy the heuristic rule. More knots are inserted into the two adjacent intervals of the feature points if the rule is not satisfied. Knot is inserted into the interval where the absolute value of the curvature difference of the endpoints is bigger than the other. As shown in Fig. 4(c), at the beginning, $|k_3^+ - k_3| > |k_3^- - k_3|$, $u_0^{'}$ is placed at the point whose curvature is nearest to $(k_3 + k_3^+)/2$. Then, more knots are inserted iteratively to make u_3 satisfy the heuristic rule. The subscript variables also indicate the order of the insertion of the knots.



(c)

Fig. 4. Knot placement: (a) curvature monotone segments; (b) curvature constant segments; (c) feature points

6. EXAMPLES

In this section we present and discuss the performance of the presented knot placement algorithm in the context of two numerical experiments. Both examples have small features that are important in defining the shape of the underlying curves. Using traditional data smoothing and reduction methods, it is very difficult to preserve them. It may take much time to process these data in practice in order to reconstruct the intended curve.

The lowpass filter, like the one shown in Fig.6, is used to smooth the discrete curvature of the data points in this paper. As discussed in section 2, the magnitude response of this filter varies smoothly in order to reduce the impact of the filter on the output. This filter is also used in both forward and reverse directions to prevent phase distortion. Refer to [1] for the implementation of digital filter in one direction.

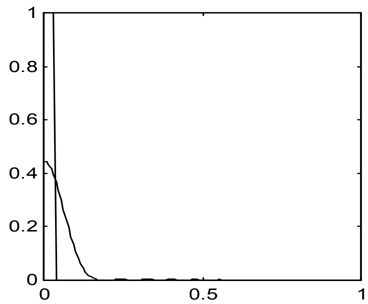
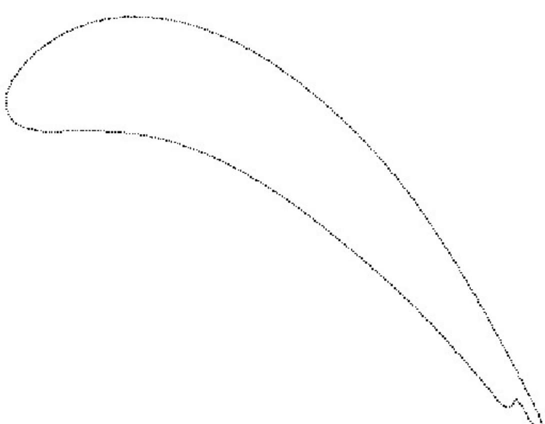


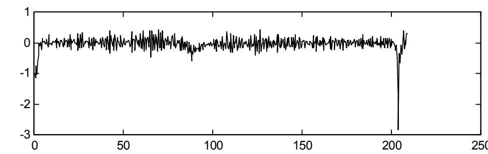
Fig. 5. Lowpass filter

The graphical output of this section consists of four kinds of figures, namely the data point figure, the curve figure, the original and smoothed discrete curvature plots of the data points. In the curve figures, the spots indicate the knots determined using the presented algorithm.

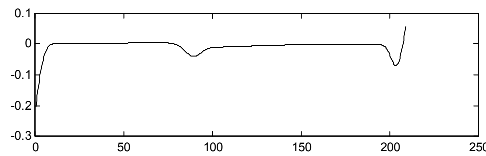
The data of the first example originate from a section of a turbine blade (see Fig. 6 (a)). The curve is sampled evenly with respect to the parameter, and the obtained data points are disturbed with number less than 0.01. The maximum and average errors are 0.07186, 0.01254, respectively. The features of the curve are retrieved successfully (see Fig. 6 (d)). The original curvature plot of the data is shown in Fig. 6(b), and the smoothed curvature plot is shown in Fig. 6(c).



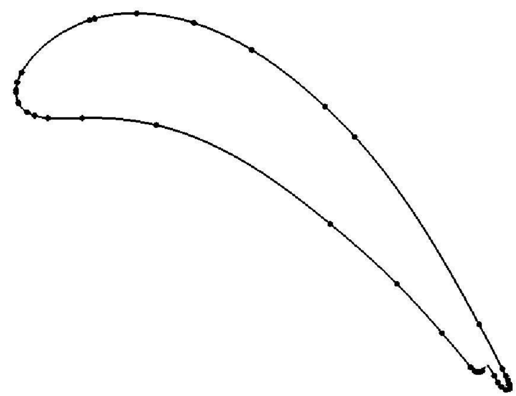
(a)



(b)



(c)



(d)

Fig. 6. Knot placement for a blade section: (a) original data points, (b) initial discrete curvature, (c) smoothed discrete curvature, (d) the approximating curve

The following example is from RE application in automotive industry. The data of this example originate from a section of the hood of a car (see Fig. 7 (a)). The points are not evenly distributed, but they are dense in most regions. From the discrete curvature plot shown in Fig.7 (b), we can observe that the noise is very severe, and it is difficult to find out the design intent from the curvature plot. The smoothed curvature shown in Fig. 7(c) mostly coincides with the design intent. The variance intention of curvature is roughly exposed. The distribution of the knots shown in Fig. 7(d) coheres with the intended variance of the curvature of the underlying curve in most portions. The features are also retrieved successfully. The maximum and average errors are 0.08616, 0.01648, respectively.

The average error of the approximating curve of the blade section data is comparable to the initial disturbance we added to the data. This proves that our algorithm is effective in retrieving the underlying curve from noisy data. The approximation error of the other example is also quite acceptable for RE applications. Meanwhile, the knots determined using our algorithm always satisfy the Schoenberg-Whitney condition. Other techniques, such as parameter correction and curve fairing [2], [6], can be used subsequently to refine the curves for the applications where the requirements are very strict.

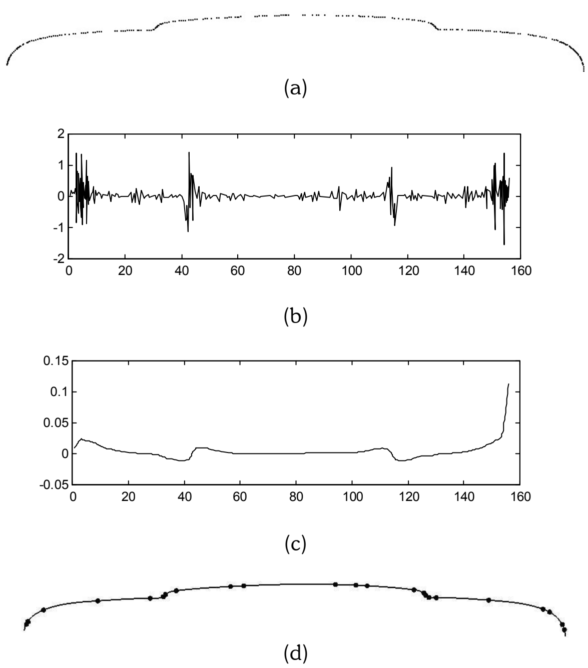


Fig. 7. Knot placement for a hood section: (a) original data points, (b) initial discrete curvature, (c) smoothed discrete curvature, (d) the approximating curve

The data points need not be pre-processed in the above knot placement procedure except the interactive deletion of outliers and dropouts when it is necessary. From the point of view of accuracy, pre-processing of the data, such as smoothing and re-sampling, may introduce uncertainty into the data. The curves reconstructed using the presented algorithm are freed from this kind of uncertainty and conform to the given data.

7. CONCLUSIONS

A new knot placement algorithm has been described in this paper for B-spline curve approximation to dense and noisy data points. In this algorithm, the discrete curvature of the data points is smoothed using lowpass digital filter to expose the curvature characteristics of the underlying curve of the data. Then knots are automatically placed to make the curve, which passes the corresponding points, be of locally small deflection. This heuristic rule for knot placement can also be used in B-spline curve approximation to smooth data points. The knots determined using this approach are sensitive to the variation of the curvature, which means knots are concentrated in the regions where the function underlying the data is more ‘severe’.

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