Subdivision Surfaces for CAD

Weiyin Ma

City University of Hong Kong, mewma@cityu.edu.hk

ABSTRACT

Subdivision surfaces refer to a class of modelling schemes that define an object through recursive subdivision starting from an initial control mesh. Similar to B-splines, the final surface is defined by the vertices of the initial control mesh. These surfaces were initially conceived as an extension of splines in modelling objects with a control mesh of arbitrary topology. They exhibit a number of advantages over traditional splines. Today one can find a variety of subdivision schemes for geometric design and graphics applications. This paper provides an overview of subdivision surfaces with a particular emphasis on schemes generalizing splines. Some common issues on subdivision surfaces modelling are addressed. Several key topics, such as scheme construction, property analysis and parametric evaluation, are discussed. Some other important topics are also summarized for potential future research and development.

Keywords: B-splines, subdivision surfaces, arbitrary topology, limit surface.

1. INTRODUCTION

In the field of computer-aided design (CAD) and related industries, the de-facto standard for shape modelling is at present non-uniform rational B-splines (NURBS). NURBS representation, however, uses a rigid rectangular grid of control points and has limitations in manipulating shapes of general topology. Subdivision surfaces provide a promising complimentary solution to NURBS. It allows the design of efficient, hierarchical, local, and adaptive algorithms for modelling, rendering and manipulating free-form objects of arbitrary topology.

In connection with shape representation, subdivision-based modeling can be dated back to Chaikin's corner cutting algorithm for defining free-form curves starting from an initial control polygon through recursive refinement [5]. In the limit, Chaikin's algorithm produces uniform quadratic B-spline curves. The scheme was later extended by Doo and Sabin [8] and Catmull and Clark [4] for defining free-form surfaces starting from an initial control mesh of arbitrary topology. For a set of regular rectangular control points, Doo-Sabin subdivision produces uniform bi-quadratic B-spline surfaces and Catmull-Clark subdivision produces uniform bi-cubic B-spline surfaces. They are therefore extensions of uniform bi-quadratic and bi-cubic B-spline surfaces, respectively, for control meshes of arbitrary topology type. In addition one can also define various sharp features, such as crease edges, corners and darts. Today, one may find rich families of subdivision surfaces (such as [4], [8-9], [13-16], [18], [23], [36]) widely used in geometric design and computer graphics for shape design, animation, multi-resolution modelling and many other engineering applications (such as [7], and see also [27-28], [39]). Subdivision surfaces possess various important properties similar to B-splines. In addition, the extension to arbitrary topology and sharp features makes subdivision surfaces a valuable asset in complimentary to NURBS.

This paper provides an introduction to subdivision surfaces with a particular emphasis on schemes that generalize B-spline surfaces.

2. THE BASIC IDEA OF SUBDIVISION

The basic idea of subdivision is to define a smooth surface as the limit surface of a subdivision process in which an initial control mesh is repeatedly refined with newly inserted vertices. Fig. 1 illustrates a closed curve refined through corner cutting using Chaikin's algorithm. Each of the control vertices of a refined mesh is computed as an affine combination of old neighboring vertices. In the limit, the refined mesh converges to a smooth curve which is known as an uniform quadratic B-spline curve.

Chaikin's subdivision has two essential components common to all subdivision schemes, i.e., topological rules and geometric rules.

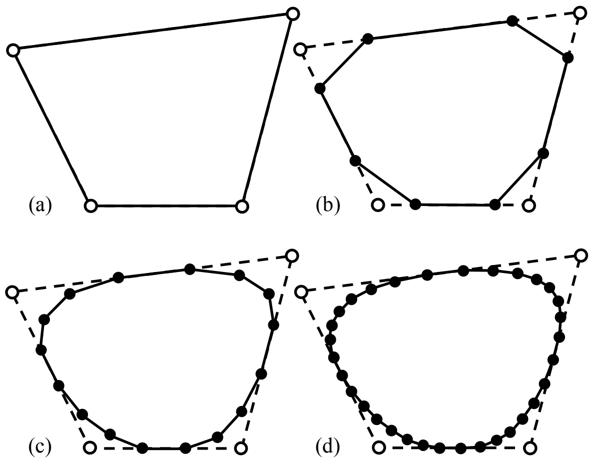

Fig. 1. Subdivision through Chaikin's corner cutting algorithm: (a) the initial control mesh; (b)-(d) control meshes after one, two and three subdivisions, respectively.

- The topological rules of Chaikin's subdivision are illustrated in Fig. 2, which is often called corner cutting. For each old vertex v_i, the corner is cut off by inserting two new vertices v'_{2i} and v'_{2i+1} and is replaced by a new edge $v'_{2i}v'_{2i+1}$ connecting the two newly inserted vertices. The length of all old edges are thus reduced, e.g., to $v'_{2i+1}v'_{2i+2}$ for old edge v_iv_{i+1}.
- The geometric rules for Chaikin's algorithm are defined by Eqn. (1), i.e., the newly inserted vertices are computed as a linear combination of old neighboring vertices. For clarity and easy implementation, Eqn. (1) is often represented by a subdivision mask as shown in Fig. 3. A newly inserted vertex (black dot) is computed as a linear combination of the old vertices (in circle). The coefficients are marked above the corresponding vertices.

$$v'_{2i} = \tfrac{1}{4}v_{i-1} + \tfrac{3}{4}v_i \tag{1a}$$

$$v'_{2i+1} = \tfrac{3}{4}v_i + \tfrac{1}{4}v_{i+1} \tag{1b}$$

3. SUBDIVISION SCHEMES FROM B-SPLINES

In literature, many subdivision schemes are further generalizations of a subset of splines. In this section, we show how the Chaikin's subdivision is constructed from quadratic B-splines. We also construct the Catmull-Clark subdivision from B-spline mathematics.

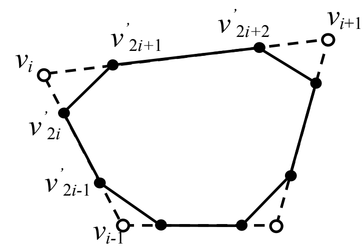

Fig. 2. Topological rules for Chinkin's subdivision.

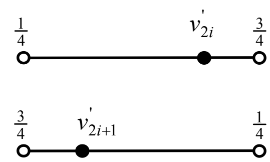

Fig. 3. Masks for Chinkin's subdivision.

3.1 Refinement of B-splines

We first examine the refinement property of B-splines. For notational simplicity, we consider a set of uniform knots $\{t_i\}_{i=-\infty}^{+\infty} = \{i\}_{i=-\infty}^{+\infty}$. All basis functions of order *k* are actually translations of the same basis function $B_k(t)$. In addition, the basis function can also be defined as a linear combination of *k*+1 translated (with index *j*) and dilated (with new parametrization 2*t*) copy of itself using the following refinement equation [39]

$$B_k(t) = \frac{1}{2^{k-1}} \sum_{j=0}^{k} \binom{k}{j} B_k(2t - j) \tag{2}$$

All subdivision schemes generalizing uniform B-splines can be derived based on Eqn. (2).

3.2 Curve subdivision scheme construction

As an example, the Chaikin's subdivision can be derived from Eqn. (2). Fig. 4 illustrates the case of order 3 and the refinement is defined as:

$$B_3(t) = \tfrac{1}{4}B_3(2t) + \tfrac{3}{4}B_3(2t-1) + \tfrac{3}{4}B_3(2t-2) + \tfrac{1}{4}B_3(2t-3) \tag{3}$$

For clarity, we illustrate all the basis functions of a uniform quadratic B-spline curve before and after mid point knot insertion in Fig. 5. Based on the refinement of Eqn. (3) and noting the notational changes in writing the refined basis functions between Fig. 4 and Fig. 5, we have

$$p(t)=\sum_{i=-\infty}^{+\infty} v_i B_{i,3}(t)$$
$$=\cdots+v_{i-1}\cdot\left(\tfrac{1}{4}B'_{2i-3,3}(t)+\tfrac{3}{4}B'_{2i-2,3}(t)+\tfrac{3}{4}B'_{2i-1,3}(t)+\tfrac{1}{4}B'_{2i,3}(t)\right)$$
$$+v_i\cdot\left(\tfrac{1}{4}B'_{2i-1,3}(t)+\tfrac{3}{4}B'_{2i,3}(t)+\tfrac{3}{4}B'_{2i+1,3}(t)+\tfrac{1}{4}B'_{2i+2,3}(t)\right)$$
$$+v_{i+1}\cdot\left(\tfrac{1}{4}B'_{2i+1,3}(t)+\tfrac{3}{4}B'_{2i+2,3}(t)+\tfrac{3}{4}B'_{2i+3,3}(t)+\tfrac{1}{4}B'_{2i+4,3}(t)\right)+\cdots \quad (4a)$$

After reorganization, we obtain

$$p(t)=\sum_{i=-\infty}^{+\infty} v_i B_{i,3}(t)$$
$$=\cdots+\left(\tfrac{3}{4}v_{i-1}+\tfrac{1}{4}v_i\right)B'_{2i-1,3}(t)+\left(\tfrac{1}{4}v_{i-1}+\tfrac{3}{4}v_i\right)B'_{2i,3}(t)$$
$$+\left(\tfrac{3}{4}v_i+\tfrac{1}{4}v_{i+1}\right)B'_{2i+1,3}(t)+\left(\tfrac{1}{4}v_i+\tfrac{3}{4}v_{i+1}\right)B'_{2i+2,3}(t)+\cdots$$
$$=\cdots+v'_{2i-1}B'_{2i-1,3}(t)+v'_{2i}B'_{2i,3}(t)+v'_{2i+1}B'_{2i+1,3}(t)+v'_{2i+2}B'_{2i+2,3}(t)+\cdots$$
$$=\sum_{i=-\infty}^{+\infty} v'_i B'_{i,3}(t) \quad (4b)$$

where v'_{2i} and v'_{2i+1} represent vertices after refinement and are defined as linear combinations of the original vertices v_{i-1}, v_i and v_{i+1} as follows:

$$v'_{2i}=\tfrac{1}{4}v_{i-1}+\tfrac{3}{4}v_i \quad (5a)$$

$$v'_{2i+1}=\tfrac{3}{4}v_i+\tfrac{1}{4}v_{i+1} \quad (5b)$$

Eqn. (5) is exactly the same as Eqn. (1), ie. the Chaikin's subdivision discussed in Section 2. The subdivision masks are illustrated in Figs. 2-3.

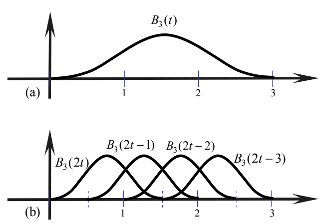

Fig. 4. Refinement of a 3rd order B-spline basis function through translation and dilation: (a) original basis function; and (b) translated and dilated copies of itself

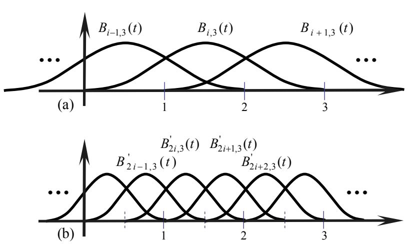

Fig. 5. Refinement of quadratic B-splines through mid-point knot insertion: (a) basis functions defined by a set of uniform knots; and (b) basis functions defined by refined knots through mid-point insertion

Following Eqn. (2), we can also refine a 4th order basis function as follows:

$$B_4(t)=\tfrac{1}{8}B_4(2t)+\tfrac{1}{2}B_4(2t-1)+\tfrac{3}{4}B_4(2t-2)+\tfrac{1}{2}B_4(2t-3)+\tfrac{1}{8}B_4(2t-4) \quad (6)$$

With similar derivation, we can also obtain the following equation for cubic B-spline curve subdivision

$$v'_{2i}=\tfrac{1}{8}v_{i-1}+\tfrac{3}{4}v_i+\tfrac{1}{8}v_{i+1} \quad (7a)$$

$$v'_{2i+1}=\tfrac{1}{2}v_i+\tfrac{1}{2}v_{i+1} \quad (7b)$$

Figs. 6-7 summarize the topological and geometric rules, respectively. For each refinement, each of the old vertices are updated and a new vertex is inserted for each edge. The subdivision masks are illustrated in Fig. 7.

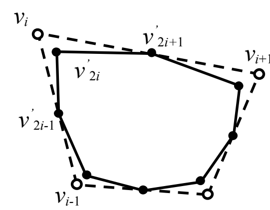

Fig. 6. Topological rules for cubic spline curve subdivision.

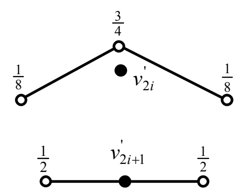

Fig. 7. Masks for cubic spline curve subdivision: (a) masks for updated corner vertices; and (b) mask for newly inserted edge vertex.

3.3 Surface subdivision scheme construction

For B-spline surfaces, we may also construct various subdivision schemes based on tensor product formulation. We take the Catmull-Clark subdivision surface as an example and construct the scheme based on extensions of refinement of uniform bi-cubic B-spline surfaces. For notational simplicity, we rewrite Eqn. (6) as follows:

$$B_{i,k}(t)=\sum_{l=0}^{k}\alpha_l B_{i,k}(2t-l)=\sum_{l=0}^{k}\alpha_l B'_{2i-k+2+l}(t)=\sum_{l=0}^{4}\alpha_l B'_{2i-2+l}(t) \quad (8)$$

with k=4 and $\{\alpha_l\}_{l=0}^{4} = \left\{\frac{1}{8}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{8}\right\}$. We have then the following refinement for bi-cubic tensor product B-spline surfaces

$$p(u,v) = \sum_{i=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty} v_{ij} B_i(u) B_j(v) = \sum_{i=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty} v_{ij} B_{ij}(u,v)$$

$$\begin{aligned}
= \cdots &+ v_{i-1,j-1} \cdot \sum_{l=0}^{4} \alpha_l B'_{2(i-1)-2+l}(u) \cdot \sum_{l=0}^{4} \alpha_l B'_{2(j-1)-2+l}(v) \\
&+ v_{i,j-1} \cdot \sum_{l=0}^{4} \alpha_l B'_{2i-2+l}(u) \cdot \sum_{l=0}^{4} \alpha_l B'_{2(j-1)-2+l}(v) \\
&+ v_{i+1,j-1} \cdot \sum_{l=0}^{4} \alpha_l B'_{2(i+1)-2+l}(u) \cdot \sum_{l=0}^{4} \alpha_l B'_{2(j-1)-2+l}(v) \\
&+ v_{i-1,j} \cdot \sum_{l=0}^{4} \alpha_l B'_{2(i-1)-2+l}(u) \cdot \sum_{l=0}^{4} \alpha_l B'_{2j-2+l}(v) \\
&+ v_{i,j} \cdot \sum_{l=0}^{4} \alpha_l B'_{2i-2+l}(u) \cdot \sum_{l=0}^{4} \alpha_l B'_{2j-2+l}(v) \\
&+ v_{i+1,j} \cdot \sum_{l=0}^{4} \alpha_l B'_{2(i+1)-2+l}(u) \cdot \sum_{l=0}^{4} \alpha_l B'_{2j-2+l}(v) \\
&+ v_{i-1,j+1} \cdot \sum_{l=0}^{4} \alpha_l B'_{2(i-1)-2+l}(u) \cdot \sum_{l=0}^{4} \alpha_l B'_{2(j+1)-2+l}(v) \\
&+ v_{i,j+1} \cdot \sum_{l=0}^{4} \alpha_l B'_{2i-2+l}(u) \cdot \sum_{l=0}^{4} \alpha_l B'_{2(j+1)-2+l}(v) \\
&+ v_{i+1,j+1} \cdot \sum_{l=0}^{4} \alpha_l B'_{2(i+1)-2+l}(u) \cdot \sum_{l=0}^{4} \alpha_l B'_{2(j+1)-2+l}(v) + \cdots
\end{aligned} \tag{9}$$

After expanding the terms and reorganization following the refined basis functions, we obtain

$$\begin{aligned}
p(u,v) &= \sum_{i=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty} v_{ij} B_{ij}(u,v) \\
&= \cdots + v'_{2i,2j} B'_{2i,2j}(u,v) + v'_{2i+1,2j} B'_{2i+1,2j}(u,v) \\
&\quad + v'_{2i,2j+1} B'_{2i,2j+1}(u,v) + v'_{2i+1,2j+1} B'_{2i+1,2j+1}(u,v) + \cdots
\end{aligned} \tag{10a}$$

where

$$\begin{aligned}
v'_{2i,2j} &= \tfrac{1}{64}\big(v_{i-1,j-1} + 6v_{i,j-1} + v_{i+1,j-1} + 6v_{i-1,j} \\
&\qquad + 36v_{i,j} + 6v_{i+1,j} + v_{i-1,j+1} + 6v_{i,j+1} + v_{i+1,j+1}\big) \\
v'_{2i+1,2j} &= \tfrac{1}{16}\big(v_{i,j-1} + v_{i+1,j-1} + 6v_{i,j} + 6v_{i+1,j} + v_{i,j+1} + v_{i+1,j+1}\big) \\
v'_{2i,2j+1} &= \tfrac{1}{16}\big(v_{i-1,j} + v_{i-1,j+1} + 6v_{i,j} + 6v_{i,j+1} + v_{i+1,j} + v_{i+1,j+1}\big) \\
v'_{2i+1,2j+1} &= \tfrac{1}{4}\big(v_{i,j} + v_{i+1,j} + v_{i,j+1} + v_{i+1,j+1}\big)
\end{aligned} \tag{10b}$$

are the refined control vertices. Following Eqn. (10), the refined control vertices can be classified into three classes, i.e., newly inserted face vertices (F-vertices) $F_{2i+1,2j+1} = v'_{2i+1,2j+1}$, updated vertex vertices (V-vertices) $V_{2i,2j} = v'_{2i,2j}$, and two newly inserted edge vertices (E-vertices) $E_{2i+1,2j} = v'_{2i+1,2j}$ and $E_{2i,2j+1} = v'_{2i,2j+1}$.

For extension to arbitrary control mesh, the newly inserted V-vertex $v'_{2i,2j}$ and E-vertices $v'_{2i+1,2j}$ and $v'_{2i,2j+1}$ of Eqn. (10b) can also be further reorganized based on the newly computed face vertices

$$\begin{aligned}
v'_{2i,2j} &= (1-\beta-\gamma)v_{i,j} + \frac{\beta}{N_v}\big(v_{i,j-1} + v_{i-1,j} + v_{i+1,j} + v_{i,j+1}\big) \\
&\qquad + \frac{\gamma}{N_v}\big(v'_{2i-1,2j-1} + v'_{2i-1,2j+1} + v'_{2i+1,2j-1} + v'_{2i+1,2j+1}\big) \\
v'_{2i+1,2j} &= \frac{1}{4}\big(v_{i,j} + v_{i+1,j} + v'_{2i+1,2j-1} + v'_{2i+1,2j+1}\big) \\
v'_{2i,2j+1} &= \frac{1}{4}\big(v_{i,j} + v_{i,j+1} + v'_{2i-1,2j+1} + v'_{2i+1,2j+1}\big)
\end{aligned} \tag{11}$$

where, $N_v = 4$ stands for the valence of the control mesh at vertex v_{ij}, $\beta = \gamma = 1/N_v$ are two constants, and

$$\begin{aligned}
v'_{2i-1,2j-1} &= \tfrac{1}{N_f}\big(v_{i-1,j-1} + v_{i-1,j} + v_{i,j-1} + v_{i,j}\big) \\
v'_{2i-1,2j+1} &= \tfrac{1}{N_f}\big(v_{i-1,j} + v_{i-1,j+1} + v_{i,j} + v_{i,j+1}\big) \\
v'_{2i+1,2j-1} &= \tfrac{1}{N_f}\big(v_{i,j-1} + v_{i,j} + v_{i+1,j-1} + v_{i+1,j}\big) \\
v'_{2i+1,2j+1} &= \tfrac{1}{N_f}\big(v_{i,j} + v_{i,j+1} + v_{i+1,j} + v_{i+1,j+1}\big)
\end{aligned} \tag{12}$$

are the updated face vertices with $N_f = 4$ being the number of vertices of the corresponding face. Now further let $\{p_k\}_{k=0}^{N_v-1} = \{v_{i,j-1}, v_{i-1,j}, v_{i+1,j}, v_{i,j+1}\}$ be a collection of edge vertices incident to v_{ij} and let $\{q_k\}_{k=0}^{N_v-1} = \{v'_{2i-1,2j-1}, v'_{2i-1,2j+1}, v'_{2i+1,2j-1}, v'_{2i+1,2j+1}\}$ $= \{F_{2i-1,2j-1}, F_{2i-1,2j+1}, F_{2i+1,2j-1}, F_{2i+1,2j+1}\}$ be a collection of newly inserted face vertices (F-vertices) incident to vertex v_{ij}. Eqn. (10) can then be represented in another form in terms of newly computed face vertices as:

$$\begin{aligned}
p(u,v) &= \sum_{i=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty} v_{ij} B_{ij}(u,v) \\
&= \cdots + v'_{2i,2j} B'_{2i,2j}(u,v) + v'_{2i+1,2j} B'_{2i+1,2j}(u,v) \\
&\quad + v'_{2i,2j+1} B'_{2i,2j+1}(u,v) + v'_{2i+1,2j+1} B'_{2i+1,2j+1}(u,v) + \cdots
\end{aligned} \tag{13a}$$

where

$$\begin{aligned}
v'_{2i,2j} &= \left((1-\beta-\gamma)v_{i,j} + \tfrac{\beta}{N_v}\sum_{i=0}^{N_v-1} p_i + \tfrac{\gamma}{N_v}\sum_{i=0}^{N_v-1} q_i\right) \\
v'_{2i,2j+1} &= \tfrac{1}{4}\big(v_{i,j} + v_{i,j+1} + F_{2i-1,2j+1} + F_{2i+1,2j+1}\big) \\
v'_{2i+1,2j} &= \tfrac{1}{4}\big(v_{i,j} + v_{i+1,j} + F_{2i+1,2j-1} + F_{2i+1,2j+1}\big) \\
v'_{2i+1,2j+1} &= \tfrac{1}{N_f}\big(v_{i,j} + v_{i,j+1} + v_{i+1,j} + v_{i+1,j+1}\big)
\end{aligned} \tag{13b}$$

are refined control vertices. Eqn. (13) establishes a subdivision scheme for uniform bi-cubic B-spline surfaces and it can be easily extended to construct a subdivision scheme for refining control meshes of arbitrary topology. One can compute newly inserted F- vertices as an average of all old vertices of the corresponding face for arbitrary N_f and then use Eqn. (13) with a general valence N_v. This leads to the following well known Catmull-Clark subdivision, i.e., a generalization of the mid-point knot insertion refinement of uniform bi-cubic B-spline surfaces shown in Figs 8-9.

- F-vertices: A face vertex for each face is computed as an average of all old control vertices of the corresponding face.
- E-vertices: An edge vertex for each edge is computed as an average of the two end vertices of the corresponding edge and the two newly inserted F-vertices whose faces share the same corresponding edge, i.e. an average of four related vertices.
- V-vertices: A vertex vertex is computed as a linear combination of the corresponding old vertex, all old vertices incident to the corresponding vertex though edges, and all newly inserted face vertices whose faces incident to the corresponding vertex.

In addition, subdivision rules for sharp feature, such as crease edges, boundary edges, corners where three or more creases meet, and darts where a crease edge terminates, can also be defined. We may keep all corners unchanged during the refinement (mask not shown in Fig. 9), use the subdivision rules for cubic curves defined by Eqn. (7) and Figs. 6-7 for refining crease and boundary edges, and use the same mask as that for smooth vertices for dart vertices. Fig. 8 illustrates how exactly the refined control mesh is constructed. Fig. 9 shows the masks for Catmull-Clark subdivision in case of general topology.

In case of extraordinary vertices whose valence is other than 4, i.e. $N_f \neq 4$, the coefficients β and γ in Eqn. (13) can be selected from a variety of ranges and can be optimized for obtaining well behaved surface properties at the extraordinary position.

In a similar way, one may also develop a tensor product version subdivision scheme for quadratic B-spline surfaces based on the Chaikin's algorithm and extend it to the welknown Doo-Sabin subdivision surfaces. Fig. 10 shows an example of a smooth Catmull-Clark subdivision surface model. Fig. 11 illustrates a pipe model and a gun model produced using Doo-Sabin and Catmull-Clark subdivision surfaces, respectively.

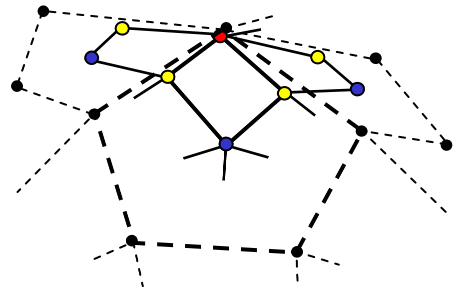

Fig. 8. Topological rules for Catmull-Clark subdivision surfaces.

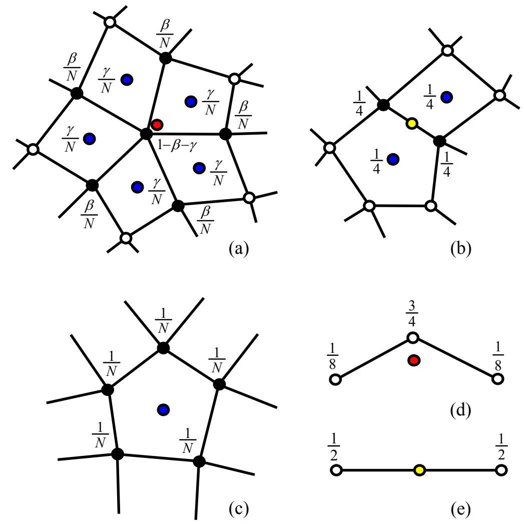

Fig. 9. General subdivision masks for Catmull-Clark surfaces: (a) mask for updated smooth V-vertices and darts; (b) mask for newly inserted E-vertices; and (c) mask for newly inserted face vertices.

4. OVERVIEW OF SUBDIVISION SCHEMES

In literature, one may find rich families of subdivision schemes. While most of the reported schemes are further generalizations of a subset of splines, some other subdivision schemes are discrete versions and extensions of other functions. There are also several other subdivision schemes whose analytic version do not exists or is not known at the moment.

4.1 Key concepts and a brief overview

As discussed in the previous section, a subdivision scheme is defined by a set of topological rules and geometric rules for mesh refinement. Topological rules define how a control mesh is split into a refined mesh. Depending on the type of a subdivision scheme, typical operations of topological rules include insertion of new vertices into edges or faces, updating of old vertices, connection of newly inserted and updated vertices (with also old ones if applicable), and removal of some vertices, edges or faces. Geometric rules are used to compute the exact coordinates of the refined control mesh. When designing geometric rules for mesh subdivision, key properties need to be considered include affine invariance, finite support with small subdivision masks, symmetry, and behaviour of the limit surface. Techniques for series analysis, such as eigen structure analysis, z-transformtion and Fourier transformation, are often used to guide the selection of appropriate subdivision masks. The following is a list of some other important concepts:

- Approximatory versus interpolatory: If the limit surface of a subdivision scheme does not go through the initial control points, the subdivision scheme is called an approximatory subdivision scheme. Examples of approximatory subdivision schemes include Loop subdivision [18], Doo-Sabin [8] and Catmull-Clark [4] subdivision surfaces. Otherwise, the scheme is an interpolatory subdivision scheme. Typical examples include Butterfly, $\sqrt{2}$ and $\sqrt{3}$ subdivisions [9, 14, 16].
- Stationary versus non-stationary subdivision: If the subdivision rules do not change during the subdivision process, the scheme is called a stationary subdivision scheme and, otherwise, a non-stationary subdivision scheme. Most of the existing subdivision schemes are stationary subdivision schemes. To produce certain classes of shapes, such as a perfect circle, a non-stationary subdivision scheme may need to be used.
- Uniform versus non-uniform subdivision: Most of the existing subdivision schemes are uniform subdivision schemes by which an existing mesh is refined uniformly through mid-point knot insertion over the entire surface for all levels of subdivision. Otherwise, it is called a non-uniform subdivision scheme. Most of the existing subdivision schemes are uniform subdivision schemes. The NURSS subdivision scheme [30] can however perform parametrized and non-uniform subdivision.
- Global versus local or adaptive subdivision: Most of the existing subdivision schemes are designed to perform global subdivision. In certain situations, a local and adaptive subdivision might be desirable. However, there are no existing subdivision schemes that can do adaptive subdivision without affecting the limit surface.

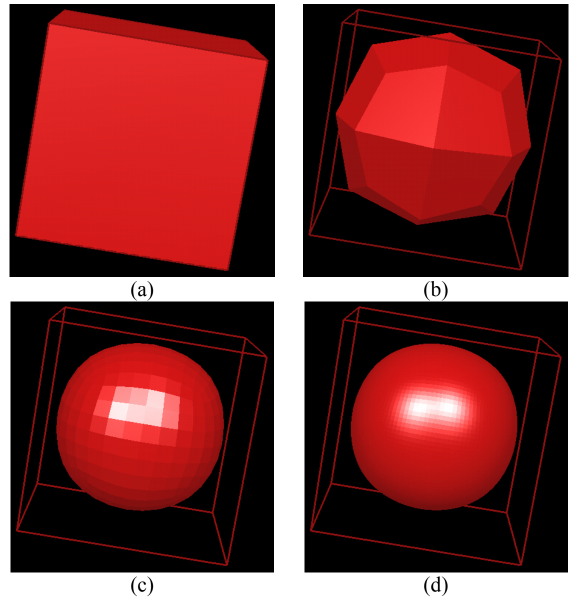

Fig. 10. Illustration of a Catmull-Clark subdivision surface: (a) the initial control mesh; (b) the control mesh after one level of refinement; (c) the control mesh after three refinement; and (d) the limit surface.

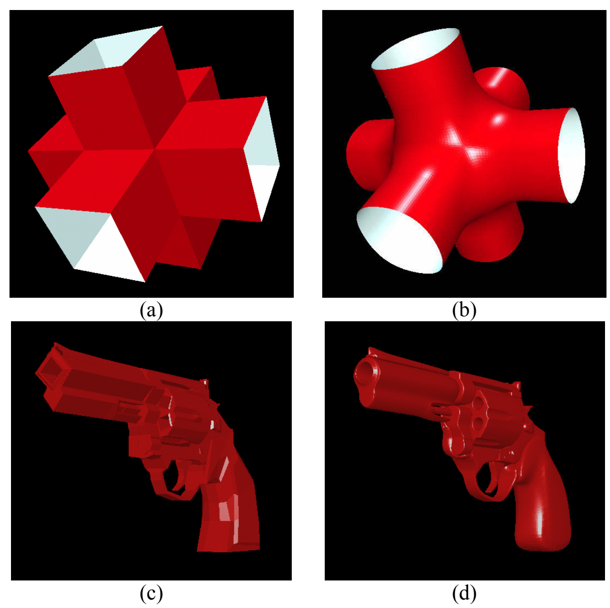

Fig. 11. Illustration of a pipe model (a)-(b) and a gun model (b)-(c) defined as a Doo-Sabin and Catmull-Clark surface, respectively. (a) and (c) initial control mesh, and (b) and (d) final limit surfaces.

Many of the existing subdivision schemes can handle sharp features, such as crease and boundary edges [3, 12, 29, 30]. Sharp features can be classified according to the number of vertices meeting at a vertex and the type of a vertex or an edge.

- Edge classification: We may distinguish three types of patch boundaries, i.e., an internal smooth edge where the limit surface is at least C1, a crease edge where the limit surface is C0, and a boundary edge where the surface terminates.
- Vertex classification: Let s be the number of crease edges meeting at a vertex. One can classify vertices into the following types according to the number of meeting crease edges s of the corresponding vertex.
    - ◊ A smooth vertex where the limit surface is at least C1 with with $s=0$.
    - ◊ A sharp vertex with $s=0$, but the limit surface is not smooth at the vertex position. If the directional tangent of the limit surface at the vertex position does not vanish, the vertex is classified as a cone-type vertex. Otherwise, if the directional tangent of the limit surface at the vertex position vanish to a single vector, it is classified as a cusp vertex.
    - ◊ A dart vertex is one where a crease edge terminates with $s=1$.
    - ◊ A crease or boundary vertex is located on a crease or boundary edge, respectively, with s=2. A boundary vertex may also be defined as a corner vertex if the boundary curve is C0 and the surface goes through that vertex.
    - ◊ A corner vertex has s≥3.

When handling sharp features, such as those for Catmull-Clark surfaces defined by the masks of Fig. 9(d)-(e), special rules need to be defined.

4.2 Properties of subdivision schemes

The analysis of subdivision surfaces at extraordinary corner or patch positions differ from that for regular parts of the control mesh. The later can often be deduced from the theory of the counter part of the scheme in continuous space, if available. For Catmull-Clark surfaces, e.g., the properties and continuity conditions of the limit surface on domains of regular grid of control points can be deduced from cubic B-spline surfaces, which is C2 continuous. Otherwise, limit surface properties can be analyzed using the same techniques as that for the analysis at extraordinary corner positions (e.g., [2, 6, 10]. At extraordinary corner positions, properties of the limit surface can be studied using various tools for series analysis, such as z-transformation, Fourier transformation, and direct eigen structure analysis of the subdivision matrix of a small invariant stencil, i.e., a subset of the control mesh, of the corresponding subdivision scheme. The analysis of subdivision schemes near extraordinary corner positions was first addressed by Doo and Sabin in [8]. The properties were then studied by Ball and Storry in [1-2] and by Sabin in [26]. Further investigations were also carried out by Reif in [25] and by Peters and Reif in [22]. One may also find some recent studies, such as [6, 24, 33, 37-38]. At the moment, an elegant theoretical foundation has been established for the analysis of various properties, such as continuity conditions and surface interrogations, of subdivision surfaces.

We again use Catmull-Clark surface to illustrate how the subdivision matrix can be set up and used for limit surface analysis, but the approach is the same for all stationary subdivision schemes. Fig. 12 illustrates such a stencil for Catmull-Clark surface before and after refinement. In the limit, the stencil converges to a point on the limit surface corresponding to v_0. Let $v^j = \left[v_0^j, v_1^j, \cdots, v_{n-1}^j\right]^T$ and $v^{j+1} = \left[v_0^{j+1}, v_1^{j+1}, \cdots, v_{n-1}^{j+1}\right]^T$ be a collection of the control points of the stencil before and after subdivision, respectively. The subdivision equation can be defined as

$$\begin{bmatrix} v_0^{j+1} \\ v_1^{j+1} \\ \vdots \\ v_{n-1}^{j+1} \end{bmatrix} = S \cdot \begin{bmatrix} v_0^{j} \\ v_1^{j} \\ \vdots \\ v_{n-1}^{j} \end{bmatrix} \tag{14}$$

or $v^{j+1} = S \cdot v^j$ in short, with S being the subdivision matrix. Various properties of the limit surface at v_0 can be determined through eigen structure analysis of the subdivision matrix S. The eigen analysis of the subdivision matrix, in turn, is also critical for designing well behaved subdivision masks, i.e., to carefully select subdivision masks and coefficients that lead to desired eigen structures and consequently well-behaved surface properties.

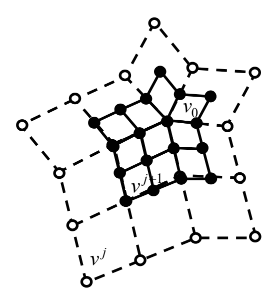

Fig. 12. Stencil for Catmull-Clark surface analysis

Let us now assume that the subdivision matrix S has real eigen values and eigen vectors $\{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$

and $\{x_0, x_1, \cdots, x_{n-1}\}$, respectively, with eigen values organized in decreasing order $\lambda_i \geq \lambda_{i+1}$. The following summarizes several important conclusions regarding the properties of subdivision surfaces in relation to the eigen structure of the subdivision matrix [11, 25, 37-38]:

- Affine invariance: The subdivision scheme is affine invariant if and only if $\lambda_0 = 1$.
- Limit position evaluation: A subdivision scheme converges if and only if $1 = \lambda_0 > \lambda_1$. Otherwise, the subdivision scheme would diverge if $\lambda_0 > 1$ and the control point/mesh would shrink to the origin if $\lambda_0 < 1$. The corresponding limit position of the control vertex v_0 is defined by $v_0^\infty = x_0^T v^0$. The tangent vectors at the limit position are defined by $c_1 = x_1^T v^0$ and $c_2 = x_2^T v^0$. The surface normal is defined by $n = c_1 \times c_2$.
- C1 continuity: The corresponding limit position of the control vertex v_0 is C1 continuous provided that (a) the characteristic map of the subdivision is regular and injective, and (b) the sub-dominant eigenvalues satisfy $1 = \lambda_0 > \lambda_1 = \lambda_2 > \lambda_3 \cdots$ or preferably $1 = \lambda_0 > \frac{1}{2} = \lambda_1 = \lambda_2 > \lambda_3 \cdots$.
- Bounded curvature: In addition to the above C1 condition, the sub-sub-dominant eigenvalue should also satisfy $\lambda_2 > \frac{1}{4} = \lambda_3 = \lambda_4 = \lambda_5 > \lambda_6 \cdots$ for obtaining bounded curvature at the limit position.

The characteristic map of an *n*-valence vertex is defined as the planar limit surface whose initial control net is defined by two eigenvectors x_1 and x_2 corresponding to the two subdominant eigenvalues λ_1 and λ_2, respectively. The *x*- and *y*- coordinates of a vertex, say v_i, of the initial control net come from the corresponding *i*-th element of the eigenvectors x_1 and x_2, respectively, while the *z*-coordinates of all the vertices are set to zero.

Note that most of the subdivision schemes, such as the original Catmull-Clark surfaces discussed in Section 3, only achieve C1 continuity at extraordinary corner positions.

4.3 Parametric evaluation

Following the discussions of last section, it is possible to evaluate the corresponding limit position, the tangent vectors and surface normal of a control vertex at any resolution in a single step. In addition, there also exists an explicit analytical form for parametric evaluation of subdivision surfaces at an arbitrary position without going through the process of infinitive subdivision. Following the derivation reported in [31] for Catmull-Clark surfaces, such a parametric evaluation exists for all stationary subdivision schemes whose regular parts are extensions of a known form in the continuous domain, which is true for most of the existing subdivision schemes found in literature. Although it has not be reported till the moment, some kind of parametric evaluation might also exist for other stationary and non-stationary subdivision schemes and might be derived based on the analysis of the subdivision scheme through various approaches for analyzing the subdivision series.

Let us take the Catmull-Clark surface [31] as an example. For regular patches, the surface can be evaluated based on uniform bi-cubic B-spline surfaces as follows:

$$p(u,v) = \sum_{i=0}^{15} v_i B_i(u,v) \tag{15}$$

where $\{B_i(u,v)\}_{i=0}^{15} = \left\{\{B_i(u) \cdot B_j(u)\}_{j=0}^{3}\right\}_{i=0}^{3}$ are the usual basis functions for uniform cubic B-spline surfaces. For extraordinary corner patches, such as the shaded patch of the initial control mesh shown in Fig. 12, the parametric form for surface evaluation is similar to Eqn. (15) with another set of basis functions $\{\psi_i(u,v)\}_{i=0}^{K-1}$ as follows:

$$p(u,v) = \sum_{i=0}^{K-1} v_i \psi_i(u,v) \tag{16}$$

where $K = 2N + 8$ is the total number of control points for an extraordinary corner patch with valance *N*. Further details for defining the basis functions $\{\psi_i(u,v)\}_{i=0}^{K-1}$ can be found in [19,31].

5. OTHER TOPICS IN SUBDIVISION MODELLING

During the past decade or so, subdivision surfaces have received extensive attention in free form surface modelling, multi-resolution representation, and computer graphics. Families of new subdivision schemes were proposed, many new theoretical tools were developed and various practical results were obtained [26-27, 35, 39]. Recently, several other topics have also attracted enormous attention and there is a lot of space for further development in many other topics.

- Unified subdivision schemes and standardization: One of the topics addressed in recent years is the pursuit of unified subdivision schemes (such as [21, 32, 34, 40]), an important step towards wide practical applications in animation, CAD and engineering in general. Ultimately a further unified generalization like NURBS for the CAD and graphics community and further standardization are expected. Such a unified generalization should cover all what we can do with NURBS, including the exact definition of regular shapes such as sphere, cylinder, cone, and various general conical shapes and rotational geometry.

- Continuity conditions at extraordinary corner positions: Another topic is the lifting of continuity conditions at extraordinary corner positions in handling general degrees that should be the same as that for regular part of subdivision surfaces. While we are striving to seek subdivision schemes with small masks for simplicity, it should be acceptable for practical use as long as it is easy to implement.
- Manipulation tools: For wide practical use, advanced manipulation tools, such as trimming, intersection, offsetting, Boolean operations, visual effects must be developed. While there have be some attempts (such as [17], [20]] there is huge space for further development in these areas.
- Other topics: Other important topics include further development in surface fitting and interpolation, faring subdivision surface generation, subdivision surface modeling from curve nets, mass property evaluation, geometry compression, interfacing issues and compatibility with existing parametric surface software, adaptive subdivision algorithms that lead to the same limit surface.

Readers are also directed to [27-28, 35, 39] for discussions on some other topics in subdivision based modeling.

6. CONCLUSIONS

Similar to spline surfaces, subdivision surfaces are defined by a set of control points. Instead of explicit parametrization for defining B-spline surfaces, the parametrization of a subdivision surface is implicitly defined by its subdivision rules, i.e. topological rules for mesh refinement and geometric rules for computing vertices of refined meshes. While a subdivision surface is defined as the limit of recursive subdivision and refinement, algorithms for explicit parametric evaluation of subdivision surfaces in a single step may also be developed. Such a parametric evaluation has been reported for Catmull-Clark and Loop surfaces, but the technique can be applied to any stationary subdivision schemes whose parametric form exists for regular part of the surface. It might also be possible to develop an explicit form for parametric evaluation of other subdivision schemes through the analysis of the subdivision series for some other subdivision schemes. Similar techniques for subdivision surface fitting and interpolation to that used for B-splines can also be developed. As a matter of fact, many approaches have been reported so far for subdivision surface fitting. The most important merit of subdivision surfaces for the CAD and graphics community is the ability in handling control meshes of arbitrary topology. In addition, subdivision algorithms, if implemented properly, can form the basis for a wide range of extremely fast and robust interrogations.

7. ACKNOWLEDGEMENTS

The work presented in this paper was supported by City University of Hong Kong through a Strategic research Grant (7001296) and by the Research Grants Council of Hong Kong through a CERG research grant (CityU 1131/03E). The control mesh of the gun model (Fig. 11(c)) was downloaded from http://www.elysium.co.jp/3dtiger/en/download/index.html with thanks.

8. REFERENCES

[1] Ball, A. A. and D. J. T. Storry, Recursively generated B-spline surfaces, *Proc. CAD 84*, 1984, pp. 112-119.

[2] Ball, A. A. and D. J. T. Storry, Conditions for tangent plane continuity over recursively generated B-spline surfaces, *ACM Trans. Graph.*, Vol. 7(2), 1988, pp. 83-102.

[3] Biermann, H., I. Martin, D. Zorin and F. Bernardini, Sharp features on multiresolution subdivision surfaces, *Proc. of Pacific Graphics 2001*, Tokyo, October 16-18, 2001.

[4] Catmull, E. and J. Clark, Recursively generated B-spline surfaces on arbitrary topological meshes, *Comp.-Aided Des.*, Vol. 10, 1978, pp. 350-355.

[5] Chaikin, G., An algorithm for high-speed curve generation, *Comp. Graphics and Image Processing*, Vol. 3, 1974, pp. 346-349.

[6] Daubechies, I., I. Guskov and W. Sweldens, Regularity of irregular subdivision, *Constr. Appr.*, Vol. 15(3), 1999, pp. 381-426.

[7] DeRose, T., M. Kass and T. Truong, Subdivision surfaces in character animation, *Proc. ACM SIGGRAPH Comp. Graphics*, 1998, pp. 85-94.

[8] Doo, D. and M. Sabin, Behaviors of recursive division surfaces near extraordinary points, *Comp.-Aided Des.*, Vol. 10, 1978, pp. 356-360.

[9] Dyn, N., D. Levin and J. A. Gregory, A butterfly subdivision scheme for surface interpolation with tension control, *ACM Trans. Graph.*, Vol. 9, No. 2, April 1990, pp. 160-169.

[10] Dyn, N., Subdivision schemes in computer aided geometric design, *Advances in numerical analysis II, Subdivision algorithms and radial functions*, W.A. Light (ed.), Oxford Univ. Press, 1992, pp. 36-104.

[11] Halstead, M., M. Kass and T. DeRose, Efficient fair interpolation using Catmull-Clark surfaces, *ACM SIGGRAPH Comp. Graphics*, 1993, pp. 35-44.

[12] Hoppe, H., T. DeRose, T. Duchamp, M. Halstead, H. Jin, J. McDonald, J. Schweitzer and W. Stuetzle, Piecewise smooth surface reconstruction, *Proc. ACM SIGGRAPH Comp. Graphics*, 1994, pp. 295-302.

[13] Kobbelt, L., $\sqrt{3}$ -subdivision,*Proc. ACM SIGGRAPH Comp. Graphics*, 2000, pp. 103-112.

[14] Labisk, U. and G. Greiner, Interpolatory $\sqrt{3}$ -Subdivision, *Comp. Graphics Forum,* Vol. 19, No. 3, 2000, pp. 131-138.

[15] Li, G., W. Ma and H. Bao, $\sqrt{2}$ Subdivision for quadrilateral meshes, *The Visual Computer*, Vol. 20, Nos. 2-3, May 2004, pp. 180-198.

[16] Li, G., W. Ma and H. Bao, Interpolatory $\sqrt{2}$ -subdivision surfaces, Proc. of GMP 2004: Geometric Modelling and Processing, IEEE Computer Press, California, 2004, pp. 185-194.

[17] Litke, N., A. Levin and P. Schröder, Trimming for subdivision surfaces. *Comp. Aided Geom. Des.*, Vol. 18, 2001, pp. 463–481.

[18] Loop, C., *Smooth subdivision surfaces based on triangles*, Master's Thesis, Department of Mathematics, University of Utah, 1987.

[19] Ma, W. and N. Zhao, Smooth multiple B-spline surface fitting with catmull-clark surfaces for extraordinary corner patches, *The Visual Computer*, Vol. 18(7), 2002, pp. 415-436.

[20] Nasri, A. H., Interpolating meshes of boundary intersecting curves by subdivision surfaces, *The Visual Computer*, Vol. 16(1), 2000, pp. 3-14.

[21] Oswald, P. and P. Schröder, Composite Primal/Dual-Subdivision Schemes, *Comp. Aided Geom. Des.*, 20(2) , 2003, pp. 135-164.

[22] Peters, P. and U. Reif, The simplest subdivision scheme for smoothing polyhedra, *ACM Trans. on Graphics*, Vol. 16, 1997, No. 4, pp. 420-431.

[23] Peters, J. and U. Reif, Analysis of algorithms generalizing B-spline subdivision, *SIAM J. on Num. Anal.*, Vol. 35(2), 1998, pp. 728-748.

[24] Prautzsch, H., Smoothness of subdivision surfaces at extraordinary points, *Adv. Comp. Math.*, Vol. 14, 1998, pp. 377-390.

[25] Reif, U., A unified approach to subdivision algorithms near extraordinary vertices, *Comp. Aided Geom. Des*, Vol. 12(2), 1995, pp. 153-174.

[26] Sabin, M. A., Cubic recursive division with bounded curvature, In P. J. Laurent, A. Le Mehaute and L. L. Schumaker (eds), *Curves and Surfaces*, Acad. Press, 1991, pp. 411–414.

[27] Sabin, M,. Subdivision surfaces, In G.E. Farin, J. Hoschek, and M.S. Kim (eds.), *Handbook of Comp. Aided Geom. Des*, 2002, pp. 309-325.

[28] Sabin, M., Recent progress in subdivision-a survey. *Advances in Multiresolution for Geometric Modeling,* Springer, 2004

[29] Schweitizer, J. E., Analysis and application of subdivision surfaces, PhD Thesis, University of Washington, Seattle, 1996.

[30] Sederberg, T. W., J. Zheng, D. Sewell and M. Sabin, Non-uniform recursive subdivision surfaces, *ACM SIGGRAPH Comp. Graphics*, 1998, pp. 387-394.

[31] Stam, J., Exact evaluation of Catmull-Clark subdivision surfaces at arbitrary parameter values, *ACM SIGGRAPH Comp. Graphics*, 1998, pp. 395-404.

[32] Stam, J., On subdivision schemes generalizing uniform B-spline surfaces of arbitrary degree, *Comp. Aided Geom. Des*, Vol. 18(5), 2001, pp. 383-396.

[33] Umlauf, G., Analyzing the characteristic map of triangular subdivision schemes, *Constr. Appr.*, Vol. 16 (1) , 2000, pp. 145-155.

[34] Warren J. and Schaefer S, A factored approach to subdivision surfaces, *Comp. Graphics & Applications*, Vol. 24(3), 2004, pp. 74-81.

[35] Warren, J. and H. Weimer, *Subdivision Methods for Geometric Design, A Constructive Approach*, Morgan Kaufmann Publishers, Boston, 2002.

[36] Velho, L. and D. Zorin, 4-8 Subdivision, *Comp. Aided Geom. Des.*, 18(5), 2001, pp. 397-427.

[37] Zorin, D., Smoothness of stationary subdivision on irregular meshes, *Constr. Appr.*, Vol. 16(3), 2000, pp.359-397.

[38] Zorin, D., A method for analysis of C1-continuity of subdivision surfaces. *SIAM J. Num. Anal.*, Vol. 37(5), 2000, pp.1677-1708.

[39] Zorin, D. and P. Schröder, *Subdivision for modeling and animation*, ACM SIGGRAPH 2000 Course Notes, July 23-28, 2000.

[40] Zorin, D. and P. Schröder, A unified framework for primal/dual quadrilateral subdivision scheme, *Comp. Aided Geom. Des.*, Vol. 18(5), 2001, pp. 429-454.