Variational Formulation of the Log-Aesthetic Surface and Development of Discrete Surface Filters

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ABSTRACT

Log-aesthetic curves include the logarithmic (equiangular) spiral, clothoid, and involute curves. Although most of these are expressed only by an integral form of the tangent vector, it is possible to interactively generate and deform them, and they are expected to be utilized in industrial and graphical design. The discrete log-aesthetic filter based on the formulation of the log-aesthetic curve has successfully been introduced to avoid strong constraints on the designer's activity, to allow free design, and to embed the properties of the log-aesthetic curves for complicated shapes with both increasing and decreasing curvature. In this paper, to define the log-aesthetic surface and develop surface filters based on its formulation, we first reformulate the log-aesthetic curve with variational principles. Then, we propose several new functionals to be minimized for free-form surfaces and define the log-aesthetic surface. Furthermore, we propose new discrete surface filters based on the log-aesthetic surface formulation.

Keywords: log-aesthetic curve and surface, variational principle, digital filter.
DOI: 10.3722/cadaps.2012.901-914

1 INTRODUCTION

Due to developments in measurement and information technology, reverse engineering can be used to generate digital models on computers from 3-dimensional physical models in clay or wood. As free-form surfaces are frequently used for aesthetic design and their measurement data are usually huge and have errors, it is very difficult to generate high-quality digital models with smooth changes of curvature from such data.

The log-aesthetic curves include the logarithmic (equiangular) curve (the slope of the LCG: logarithmic curvature graph $\alpha = 1$), the clothoid curve ($\alpha = -1$), the circle involute ($\alpha = 2$), and Nielsen's spiral ($\alpha = 0$). Recently the generalized Cornu spiral [12] has been reported to include several log-aesthetic curves as its curvature profile is given by a rational linear function and so its LCG gradient is given by a straight line function [11]. It is possible to generate and deform the log-aesthetic curve in real-time even if expressed by integral forms using the unit tangent vectors as integrands ($\alpha \neq 1, 2$) and they are expected to be useful in practical applications [1, 20]. Furthermore, Ziatdinov et al. [28] recently showed that the log-aesthetic curve can be parametrically expressed in terms of incomplete gamma functions, which gives an exact analytic representation of a curve segment for any real value of α and the computation time for generating a log-aesthetic curve segment using the incomplete gamma functions is about 10 times faster than that using direct numerical integration. The discrete log-aesthetic filter based on the formulation of the log-aesthetic curve has successfully been introduced to avoid strong constraints on the designer's activity, to allow free design and to embed the properties of the log-aesthetic curves for complicated curves with both increasing and decreasing curvature [21].

Therefore, in this paper, we first define the log-aesthetic surface in two ways−one utilizes the self-affinity of the surface and the other is based on the variational principle. Then, we propose a discrete filter named log-aesthetic surface filter that removes noise from a set of points obtained by a 3D laser range scanner, smooths them out, and make the surface log-aesthetic.

The rest of the paper is organized as follows. Section 2 describes related work and section 3 discusses the self-affinity of the surface and the formulation of the log-aesthetic surface using it. Section 4 explains the formulation of the log-aesthetic surface based on the variational principle. Section 5 introduces a new discrete surface filter based on the log-aesthetic surface. Finally, we conclude the paper in section 6 with a discussion of future work.

2 PRELATED WORK

In this section, we discuss related research on the log-aesthetic curve, curvature based energy functionals, for fair surfaces, and discrete filters.

2.1 Log-aesthetic Curve

"Aesthetic curves" were proposed by Harada et al. [7] as curves whose logarithmic distribution diagram of curvature (LDDC) can be approximated by a straight line. Miura et al. [17, 18] derived analytical solutions of the curves whose logarithmic curvature graph (LCG) −an analytical version of the LDDC [7] is strictly given by a straight line and proposed these lines as general equations of aesthetic curves. Furthermore, Yoshida and Saito [26] analyzed the properties of the curves expressed by the general equations and developed a new method to interactively generate a curve by specifying two end points and the tangent vectors with three control points as well as the slope of the straight line of the LCG. In this research, we call the curves expressed by the general equations of aesthetic curves the "log-aesthetic curves."

The problems of the connection of multiple log-aesthetic segments was dealt with by Miura et al. [20] and an input method of the compound-rhythm log-aesthetic curve consisting of two log-aesthetic curve segments connected with C^3 continuity was proposed by Agari [1]. Furthermore, an extension of the planar log-aesthetic curve into space−the log-aesthetic space curve−was proposed by Miura et al. [19], and it was classified by Yoshida and Saito [27]. This section discusses several important properties of log-aesthetic curves. Note that an aesthetic curve is a curve whose logarithmic curvature graph is given by a straight line.

2.1.1 General equations of aesthetic curves

For a given curve, we assume the arc length of the curve and the radius of curvature are denoted by s and ρ, respectively. The horizontal axis of the logarithmic curvature graph measures $\log\rho$ and the vertical axis measures $\log(ds/d(\log\rho)) = \log(\rho\, ds/d\rho)$. If the LCG is given by a straight line, there exists a constant α such that the following equation is satisfied:

$$\log\left(\rho\frac{ds}{d\rho}\right) = \alpha\log\rho + C \tag{2.1}$$

where C is a constant. The above equation is called the fundamental equation of aesthetic curves [8]. Rewriting Eqn. (2.1), we obtain:

$$\frac{1}{\rho^{\alpha-1}}\frac{ds}{d\rho} = e^{C} = C_0 \tag{2.2}$$

Hence, there is some constant c_0 such that:

$$\rho^{\alpha-1}\frac{d\rho}{ds} = c_0 \tag{2.3}$$

From the above equation, when $\alpha \neq 0$, the first general equation of aesthetic curves

$$\rho^{\alpha} = c_0 s + c_1 \tag{2.4}$$

is obtained. If $\alpha = 0$, we obtain the second general equation of aesthetic curves aesthetic curves

$$\rho = c_0 e^{c_1 s} \tag{2.5}$$

A curve that satisfies Eqn. (2.4) or Eqn. (2.5) is called a log- aesthetic curve.

2.1.2 Parametric expressions log-aesthetic curves

In this subsection, we will show parametric expressions of the log- aesthetic curves.
We assume that a curve $C(s)$ satisfies Eqn. (2.4). Then

$$\rho(s) = (c_0 s + c_1)^{\frac{1}{\alpha}} \tag{2.6}$$

As s is the arc length, $|dC(s)/ds| = 1$ (for example, refer to [5]) and there exists $\theta(s)$ satisfying the following two equations:

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta \tag{2.7}$$

Since $1/(d\theta/ds)$,

$$\frac{d\theta}{ds} = (c_0 s + c_1)^{-\frac{1}{\alpha}} \tag{2.8}$$

If $\alpha \neq 1$,

$$\theta = \frac{\alpha(c_0 s + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_0} + c_2 \tag{2.9}$$

If the start point of the curve is given by $P_0 = C(0)$,

$$C(s) = P_0 + e^{ic_2}\int_0^s e^{i\frac{\alpha(c_0 u + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_0}}\, du\theta \tag{2.10}$$

For the second general equation of aesthetic curves expressed by Eqn. (2.9),

$$\frac{d\boldsymbol{\theta}}{ds} = \frac{1}{c_0}e^{-c_1 s} \tag{2.11}$$

$$\theta = -\frac{1}{c_0 c_1} e^{-c_1 s} + c_2 \tag{2.12}$$

Therefore the curve is given by

$$C(s) = P_0 + e^{ic_2} \int_0^s e^{-\frac{i}{c_0 c_c} e^{-c_1 u}} du \tag{2.13}$$

2.2 Curvature Based Energy Functionals for Fair Curves and Surfaces

Most surface energy functionals for fair surfaces are related to energy functionals designed for curves. We can design various surface energy functionals by selecting different properties of the curves on the surface to measure curvature or change in curvature and by considering special subsets of the surface curves, i.e., geodesics, lines of curvature [13].

2.2.1 Bending energy

The bending energy functional is a generalization of Bernoulli's "elastica" energy that measures the square of curvature $\int \kappa^2 ds$ integrated over the length of a given curve. For curves on a surface, we usually consider only the normal curvature $\kappa_n(\theta)$. This curvature is a function of the principal curvature $(\kappa_{max}, \kappa_{min})$ parameterized by the angle θ made with the first principal direction. The following functional is often used for the bending energy functional E_B as an area integral of the surface,

$$E_B = \int (\kappa_{max}^2 + \kappa_{min}^2) dA \tag{2.14}$$

The above equation can be reformulated as follows:

$$E_B = \int (\kappa_{max}^2 + \kappa_{min}^2) = 4 \int H^2 dA - 2 \int G dA \tag{2.15}$$

where H and K are the mean and Gaussian curvatures, respectively. Note that the bending energy is shift-invariant because the area is expanded or shrunk at the square of the dimensional scaling factor, but the square of the curvature is inversely decreased or increased at the same rate.

2.2.2 MVS energy

Moreton and Séquin [24] introduced the "MVS" functional that measures curvature variation by integrating the squares of derivatives of the principal curvatures in their respective principal directions. Multiplication of the area term is for scale invariance [23].

$$E_{MVS} = \int (\frac{d\kappa_{max}}{de_{max}}^2 + \frac{d\kappa_{min}}{de_{min}}^2)\, dA \int dA \tag{2.15}$$

where e_{max} and e_{min} are principal curvature directions.

2.2.3 MVS_{cross} energy

Joshi and Séquin [13] introduced the "MVS_{cross}" functional that adds the change in normal curvature along the in-line direction to the MVS functional.

$$E_{MVS_{cross}} = \int (\frac{d\kappa_{max}}{de_{max}}^2 + \frac{d\kappa_{min}}{de_{min}}^2 + \frac{d\kappa_{max}}{de_{min}}^2 + \frac{d\kappa_{min}}{de_{max}}^2)\, dA \int dA \tag{2.15}$$

The MVS_{cross} energy of a surface roughly measures the deviation of the surface from a perfect sphere or a cylinder.

As discussed above, many types of functional for fair surfaces have been proposed, but for aesthetic design, designers usually do not use them for practical design because the controllability of the surface deformation is not high enough and it takes a great deal of time to minimize the functional for surfaces.

2.3 Discrete Filter

For the generation of high-quality surfaces used for car styling design, Farin et al. [6] proposed a surface smoothing method that sequentially selects a point on the curve for each character line of the surface where the curvature variation criterion introduced is the highest in the curve and locally smooth the curve around the point. In their method, a B-spline curve is first fitted to an input sequence of points. Then, such a point is extracted where the difference in the third derivative, or the derivative of the curvature is the largest and removes a knot corresponding to the point. By repeating this process, the shape of the curve can be modified to have a smooth curvature plot whose horizontal axis is the arc length and vertical axis is the curvature. Eck and Jaspert proposed a method to use the difference in curvature calculated discretely as a local criterion as a fairing method of a sequence of points without B-spline curves [4]. Wagner proposed a method to smooth trajectories of robot manipulation using fourth differences of the sequence of points instead of curvature [25]. Based on the method proposed by Wagner, Higashi and Yamada made it applicable to a curve with a non-uniform knot vector by replacing the fourth difference with the fourth divided difference and extended it to discrete surfaces that may have defect points [9, 10].

The methods mentioned above are for aesthetic design and they can yield curves and surfaces of a certain quality from the viewpoint of monotonicity of the curvature variation, or smoothness, but they do not remove the curvature instability that exists in polynomial curves like B-spline noted by Miura [16]. On the other hand, the log-aesthetic filter [21] can control the curvature. The discrete log-aesthetic filter does not minimize any integral quantities or perturb the positions of the points to minimize any objective functions. It finds locally the most approximate log-aesthetic curve for a given set of points and fits the points to the selected log-aesthetic curve. This is the main difference between methods that do and do not have desired shape targets as the log-aesthetic curve.

To generate a high-quality surface for aesthetic design, it is desirable to solve the true nonlinear minimization problem as reported by Moreton and Séquin [22] and Joshi and Séquin [13]. Schneider and Kobbelt [24] solved the nonlinear equation $\Delta H = 0$ where H is the mean curvature, and Bobenko and Schröder minimized the discrete Willmore flow [2]. Eigensatz et al. applied a bilateral filter directly to the discrete mean curvature function [5]. At present, for surface fairing it is not possible to obtain both capability of representation flexibility and high quality required for aesthetic design by minimizing objective functions based on the variational principle.

In this paper, instead of minimizing a specific functional, we propose a surface filter that makes the surface have a specified Gaussian curvature distribution in a similar way for the discrete log-aesthetic filter to give the curve a specified curvature distribution.

3 SELF-AFFINITY OF SURFACE

In this and the following section, we discuss how to extend the log-aesthetic curve formulation into a surface. Research on the log-aesthetic surface was initiated by Kanaya et al. [14], and Hadara et al. [8] proposed the log-aesthetic curved surfaces, but the formulations of such surfaces have yet to be established.

3.1 Correspondences among Differential Geometrical Quantities

As the log-aesthetic curve describes the relationship between its radius of curvature and arc length, it is necessary to specify some quantities corresponding to them to extend it into a surface. As pointed out by Miura et al. [21], among the arc length s, the curvature κ, and the arc length of its image in the indicatrix of tangents, there is such a relationship that $\kappa = \lim_{s \to 0} \sigma / s$, and it is similar to

$K = \lim_{S\to 0} S'/S$ where K is Gaussian curvature, S is the area of a surface $\boldsymbol{S}(u,v)$, and S' is the area of the Gaussian map [4]. Hence, we let the curvature of the curve κ and the arc length s correspond to the Gaussian curvature K and the surface area S, respectively. This implies that the circular arc with a constant curvature corresponds to a surface with a constant Gaussian curvature. By using these correspondences, the fundamental equation of the aesthetic curve $\rho^{\alpha-1}d\rho/ds = C_0$ corresponds to $(1-K)^{\alpha-1}d(1/K)/dS = C_1$ where C_0 and C_1 are arbitrary constants. Note that

$$\lim_{S\to 0}\frac{S'}{S} = \frac{\iint |N_u \times N_v| dudv}{\iint |S_u \times S_v| dudv} = K \tag{3.1}$$

where N_u and N_v are the derivative of the surface normal with respective to parameters u and v, respectively.

The mean curvature H is as important a surface curvature as the Gaussian curvature. For example, the surface with $H = 0$ is called a minimal surface and it is an important example in variational principle [3]. The thin membrane of soap surrounded by an arbitrary boundary is always a minimal surface. However, as the Gaussian curvature $K \le 0$, the surface cannot possess blobby parts and it is frequently inadequate for aesthetic design. In geometric modeling, for example, as mentioned in Section 1, Schneider and Kobbelt [24] solved the nonlinear equation $\Delta H = 0$. It will be necessary to research filters using the mean curvature in future.

3.2 Self-Affinity of the Plane Curve

We define self- affinity of the plane curve as follows [19]. Self-affinity of the plane curve: For a curve generated by removing an arbitrary head portion of the original curve, by scaling it with different factors in its tangent and normal directions on every point on the curve, if the original curve is obtained, then the curve has self- affinity. If a given plane curve satisfies $\rho^{\alpha-1}d\rho/ds = C$, the curve has self- similarity according to this definition [17, 18].

For a given curve $\boldsymbol{C}(s)$ parameterized by the arc length parameter s, we assume that the derivative of its curvature, hence that of its radius of curvature as well are continuous. That is, we assume the curve has C^3 continuity. In addition, the radius of curvature ρ is assumed not to be equal to 0.

By scaling the curve with different factors in the tangent and normal directions (affine transformation of the plane curve [21]), we discuss how to make the scaled curve become congruent with the original curve. Therefore, we re- parameterize the given curve $\boldsymbol{C}(s)$ using a new parameter $t = as + b$ where a and b are positive constants as shown in Fig. 1. Scaling the curve uniformly in the tangent direction is equivalent to relating a point $\boldsymbol{C}(t_0 = as_0 + b)$ to another point $\boldsymbol{C}(s_0)$ as shown in Fig.1. In this relationship the scaling factor in the tangent direction f_t is given by $1/a$.

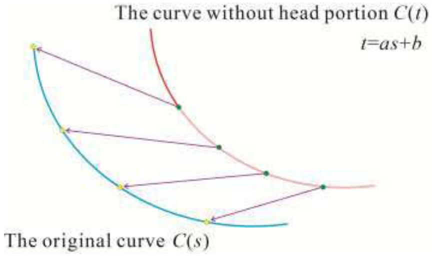


Fig. 1: Self- affinity of the plane curve.

Although a and b are constants, they are related to the scaling factors in the tangent and normal directions f_t and f_n, and depend on the shape of the curve. Hence, we cannot specify them independently. The start point of the curve $\boldsymbol{C}(t)$ is given by $\boldsymbol{C}(b)$, which is a point when $s=0$. Hence, $C(t)$ is a curve without the head portion of the original curve $\boldsymbol{C}(s)$.

The condition can be described for a curve to have self- affinity as follows. Condition for a plane curve to have self-affinity: For an arbitrary constant $b>0$, some $a>0$ is determined. With these a and b, for any $s>0$ the following equation is satisfied.

$$\frac{\rho(s)}{\rho(as+b)} = f_n \tag{3.2}$$

where f_n is a constant dependent on and determined by b and it is a scaling factor in the normal direction. f_n is given by substituting $s=0$ into the above equation as follows:

$$f_n = \frac{\rho(0)}{\rho(b)} \tag{3.3}$$

3.3 Self-Affinity of the Surface

Although the curve has arc length parameterization, the surface does not have suitable parameterization for analysis. However, it is possible to have such local parameterization that the tangent vectors with respect to the parameters are orthogonal to each other and their directions are the same as the principal direction and the norm of each tangent vector is equal to 1 [3]. Such parameterization is called isometric. In general, it is not possible to have a globally isometric parameterization. Here, we developed a discrete filer for the surface and assumed such locally isometric parameterization.

We use the reciprocal of the Gaussian curvature $R=1/K$ and define the condition for self- affinity of the surface as follows. Condition for a surface to have self-affinity: We assume that the surface is given by $\boldsymbol{S}=\boldsymbol{S}(s,t)$. For an arbitrary constant $b>0$ and $d>0$, some $a>0$ and $c>0$ are determined. With these a, b, c and d, for any $s>0$ and $t>0$, the following equation is satisfied.

$$\frac{R(s,t)}{R(as+b,cs+d)} = f_R \tag{3.4}$$

where f_R is a constant dependent on and determined by only b and d, and it does not depend on s or t. It is a scaling factor of R and is given by substituting $s=0$ and $t=0$ into the above equation as follows:

$$f_R = \frac{R(0,0)}{R(b,d)} \tag{3.5}$$

The scaling factors f_s and f_t in the directions of $\boldsymbol{S}_u$ and $\boldsymbol{S}_v$ are given by $f_s=1/a$ and $f_t=1/c$.

If $f_R=1$, from Eqn. (3.4),

$$R(s,t) = R(as+b,ct+d) \tag{3.6}$$

In this case $R(s,t)$ is constant, i.e. the Gaussian curvature K is also constant. The surface includes a planar, spherical, or cylindrical surface. Furthermore, if one of the principal curvatures is equal to 0, it can represent a developable surface, including conical and tangent surfaces. The spherical surface is the most typical type of surface of revolution with a positive constant Gaussian curvature. The surface of revolution with $K>0$ includes rugby- ball and barrel types. If $K<0$, it includes the pseudosphere [4].

If $f_R \neq 1$, then Eqn. (3.4) can be rewritten as follows:

$$R(s,t) - f_R(b,d)R(as+b,ct+d) = W(s,t) = 0 \tag{3.7}$$

Hence the function $W(s,t)$ is always equal to 0. The necessary and sufficient condition for its directional derivative to be always equal to 0 is $\nabla W(s,t)=0$. Therefore

$$\frac{\partial W(s,t)}{\partial s}=\frac{\partial R(s,t)}{\partial s}-a\ f_R\frac{\partial R(u,ct+d)}{\partial u}\bigg|_{u=as+b}=\frac{\partial R(s,t)}{\partial s}-\frac{f_R}{f_s}\frac{\partial R(u,ct+d)}{\partial u}\bigg|_{u=as+b}=0 \tag{3.9}$$

$$\frac{\partial W(s,t)}{\partial t}=\frac{\partial R(s,t)}{\partial t}-c\ f_R\frac{\partial R(as+v,v)}{\partial v}\bigg|_{v=ct+d}=\frac{\partial R(s,t)}{\partial s}-\frac{f_R}{f_t}\frac{\partial R(as+d,v)}{\partial v}\bigg|_{v=ct+d}=0 \tag{3.10}$$

Eqns. (3.9) and (3.10) are similar to the equation obtained in the curve case [19], and we obtain the following equation:

$$R(s,t)=(c_0s+c_1)^{\frac{1}{\alpha}}+(c_2t+c_3)^{\frac{1}{\beta}} \tag{3.11}$$

where $\alpha=\log f_R/\log f_s$ and $\beta=\log f_R/\log f_t$.

A similar argument for K instead of R works out and the following equation on K can be obtained

$$K(s,t)=(c_0s+c_1)^{\frac{1}{\alpha}}+(c_2t+c_3)^{\frac{1}{\beta}} \tag{3.12}$$

where $\alpha=\log f_K/\log f_s$ and $\beta=\log f_K/\log f_t$ if

$$\frac{K(s,t)}{K(as+b,cs+d)}=f_K \tag{3.13}$$

4 VARIATIONAL FORMULATION

In this section, we first discuss the variational principle with a simple example and explain how to formulate the log-aesthetic curve, especially with regard to the functional which the log-aesthetic curve minimizes. Then, we extend the functional to formulate the log-aesthetic surface.

4.1 Variational Principle

The variational analysis deals with a problem where an objective functional in an integral form should be minimized or maximized. For example,

$$J=\int_{x_1}^{x_2} f(y,y_x,x)\,ds \tag{4.1}$$

where y is a function of x and y_x is a derivative of y with respect to x. y is unknown. The condition that J has a stationary value is given by the following partial differential equation:

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y_x}=0 \tag{4.2}$$

This is called the Euler equation. If $f=f(y,y_x)$, i.e., f is given explicitly without x, the above equation means that

$$f-y_x\frac{\partial f}{\partial y_x}=c \tag{4.3}$$

where c is a constant.

The simplest example of the variational problem is to minimize the distance between two given points in the x-y plane. An infinitesimal element of the distance is given by

$$ds=\sqrt{(dx)^2+(dy)^2}\,dx \tag{4.4}$$

and the distance J is given by

$$J=\int_{x_1,y_1}^{x_2,y_2} ds=\int_{x_1}^{x_2}\sqrt{1+y_x}\,dx \tag{4.5}$$

Hence $f(y, y_x, x) = (1+y_x{}^2)^{\frac{1}{2}}$ and is given explicitly without x. By Eqn. (4.3), we obtain

$$\frac{1}{\sqrt{1+y_x^2}} = c \tag{4.6}$$

Therefore there exists a constant a such that $y_x = a$. It yields

$$y = ax + b \tag{4.7}$$

where b is a constant as well as a. These constants are determined by making the line pass through the given two points (x_1, y_1) and (x_2, y_2).

4.2 Variational Formulation of Log-aesthetic Curve

In Eqn. (2.4) if we substitute ρ^α with σ, then the equation is given by

$$\sigma = cs + d \tag{4.8}$$

The above equation means that the log- aesthetic curve is given by a straight line in the $s-\sigma$ plane where the horizontal and vertical axes are the arc length s and $\sigma = \rho^\alpha$, respectively to connect two given points (s_1, β_1) and (s_2, β_2). In this case, the following objective functional J_{LAC} is minimized.

$$J_{LAC} = \int_{s_1}^{s_2} \sqrt{1+\sigma_s^2}\, ds = \int_{s_1}^{s_2} \sqrt{1+\alpha^2 \rho^{2\alpha-2} \rho_{s\,s}^2}\, ds \tag{4.9}$$

4.3 Variational Formulation of Log-aesthetic Surface

Here, we apply the variational principle to the surface formulation. As discussed in Section 3.1, we let the curvature of the curve κ and the arc length s correspond to the Gaussian curvature K and the surface area S, respectively. In Eqn. (4.9), when $\alpha = -1$, $\kappa_s = -\rho_s / \rho^2$ and we obtain the following equation.

$$J_{LAC} = \int_{s_1}^{s_2} \sqrt{1+\kappa_s^2}\, ds \tag{4.10}$$

By reparameterizing the above equation with $s = s(t)$, it becomes

$$J_{LAC} = \int_{t_1}^{t_2} \sqrt{x_t^2 + y_t^2 + \kappa_t^2}\, dt = \int_{t_1}^{t_2} \sqrt{\lambda_C^2 + \kappa_t^2}\, dt \tag{4.11}$$

where $s_1 = s(t_1)$, $s_2 = s(t_2)$, and $\lambda_C = \sqrt{x_t^2 + y_t^2}$. Note that $ds / dt = \lambda_C$.

By extending Eqn. (4.11) into the surface, we define the objective functional for the surface J_{LAC} as follows:

$$J_{LAS} = \int_{u_1}^{u_2} \int_{v_1}^{v_2} \sqrt{\mathbf{det}(\boldsymbol{I}) + K_u^2 + K_v^2}\, du dv \tag{4.12}$$

where $\boldsymbol{I}$ is a matrix expressed with the first fundamental quantities by

$$\boldsymbol{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{4.13}$$

where $E = \boldsymbol{S}_u \cdot \boldsymbol{S}_u$, $F = \boldsymbol{S}_u \cdot \boldsymbol{S}_v$, and $G = \boldsymbol{S}_v \cdot \boldsymbol{S}_v$. Note that the area of the surface S is given by

$$S = \int_{t_1}^{t_2} \sqrt{\mathbf{det}(\boldsymbol{I})}\, du dv \tag{4.14}$$

As in Section 3.3, we assume local parameterization (s,t) around a point on the surface $\boldsymbol{S}(s_1, t_1)$ such that the tangent vectors with respect to the parameters are orthogonal to each other, their directions are the same as the principal direction, and the norm of each tangent vector is equal to 1. With this parameterization, $\boldsymbol{I}$ becomes the 2×2 unit matrix. By performing integration around the point $\boldsymbol{S}(s_1, t_1)$, Eqn. (4.11) is rewritten as

$$\Delta J_{LAS} = \int_{s_1}^{s_1+\Delta s}\int_{t_1}^{t_1+\Delta t} \sqrt{1+K_s^2+K_t^2}\,dsdt \tag{4.15}$$

According to variational principle, to minimize the following functional,

$$J = \int_{s_1}^{s_2}\int_{t_1}^{t_2} g(K, K_s, K_t, s, t)dsdt \tag{4.16}$$

the following equation should be satisfied.

$$\frac{\partial g}{\partial K} - \frac{\partial}{\partial s}\frac{\partial g}{\partial K_s} - \frac{\partial}{\partial t}\frac{\partial g}{\partial K_t} = 0 \tag{4.17}$$

Note that $g = \sqrt{1+K_s^2+K_t^2}$ does not explicitly depend on K. Eqn. (4.17) yields

$$(1+K_t^2)K_{ss} - 2K_sK_tK_{st} + (1+K_s^2)K_{tt} = 0 \tag{4.18}$$

The above equation is called the minimal surface or Lagrange's equation and the surface $\boldsymbol{S}(s,t) = (s,t,K(s,t))$ is given by a minimal surface. Therefore, in a case where the Gaussian curvature on the boundary is specified, the Gaussian curvature should be given by a minimal surface interpolating the boundary values. The above discussion assumes locally isometric parameterization and globally isometric parameterization does not exist in general. It is not possible to deal with the case where the functional is defined globally as in Eqn. (4.12). In such a case, some optimization technique should be adopted to minimize the functional to generate a target surface.

According to Bernstein's theorem [14], if the boundary of the surface is located infinitely far away, the minimal surface is given by a plane. Therefore, the Gaussian curvature is given by

$$K(s,t) = c_0 s + c_1 t + c_2 \tag{4.19}$$

where c_0, c_1, and c_2 are constants. When $\alpha = \beta = 1$ in Eqn. (3.12), the above equation is equivalent to Eqn. (3.12).

For further extension, we may use the mean curvature H instead of the Gaussian curvature K and a similar discussion is also satisfied. In this section we have not discussed the effects of the powers α and β . To take into account the effects of these powers, we may use $\kappa_{max}^{\beta}\kappa_{min}^{\beta}$ where κ_{max} and κ_{min} are the maximum and minimum normal curvatures, respectively. For example, an objective functional may be defined by

$$\begin{aligned} J_{LAS} &= \int_{u_1}^{u_2}\int_{v_1}^{v_2} \sqrt{\boldsymbol{det}(\boldsymbol{I}) + (\kappa_{max}^{\alpha}\kappa_{max}^{\beta})_u^2 + (\kappa_{max}^{\alpha}\kappa_{max}^{\beta})_v^2}\,dudv \\ &= \int_{u_1}^{u_2}\int_{v_1}^{v_2} \sqrt{\boldsymbol{det}(\boldsymbol{I}) + 2(\kappa_{max}^{2\alpha-1}\kappa_{min}^{2\beta-1})\{\alpha\kappa_{min}(\kappa_{max,u}+\kappa_{max,v}) + \beta\kappa_{\max}(\kappa_{min,u}+\kappa_{min,v})\}}\,dudv \end{aligned} \tag{4.20}$$

These extensions are topics for future research and are not dealt in this paper.

5 DISCRETE LOG-AESTHETIC SURFACE FILTER

The discrete log-aesthetic curve filter is constructed based on Eqn. (2.4) [21]. Similarly we construct a discrete surface filter based on Eqn. (3.12) for a triangular-meshed surface. Here we intend to construct an isotropic filter the effect of which does not depend on the direction on the tangent plane and we assume $\alpha = \beta$ in Eqn. (3.12).

As the simplest case, we assume $\alpha = \beta = 1$. Note that if $\alpha = \beta = 1$, Eqns. (3.12) and (4.19) are equivalent. Hence, the filter has a property to approximate the distribution of the Gaussian curvature by a plane.

We move the location of a vertex $\boldsymbol{P}_i$ in the mesh to satisfy Eqn. (4.19). We restrict the new location $\boldsymbol{P}_i'$ of $\boldsymbol{P}_i$ to satisfy

$$\boldsymbol{P}_i' = \boldsymbol{P}_{ic} + \phi\,\boldsymbol{N}_i \tag{5.1}$$

where $\boldsymbol{P}_{ic}$ is the average location of the vertices connected to $\boldsymbol{P}_i$ and $\boldsymbol{N}_i$ is a normal vector there. The coefficients c_0, c_1, and c_2 in Eqn. (4.19) are determined by projecting the positions of the vertices close to $\boldsymbol{P}_i$ to the tangent plane there by use of the values of the Gaussian curvature of these vertices by the least squares method. The value of ϕ is determined to have the value of the plane at $\boldsymbol{P}_i$.

5.1 Calculation of Mesh Curvature

The Gaussian curvature K at the vertex $\boldsymbol{P}_i$ of the triangular mesh is approximately given by

$$K = \frac{a}{S} \tag{5.2}$$

where $a = 2\pi - \sum_{j=0}^{n} \boldsymbol{\theta}_j$ and $S = \sum_{j=0}^{n} S_j / 3$ as shown in Fig. 2. The summation is performed for the triangles around the vertex $\boldsymbol{P}_i$ and $\boldsymbol{\theta}_j$ is an angle of the j-th triangle there and S_j is its area.

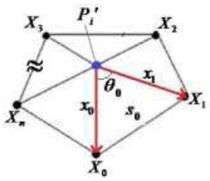


Fig. 2: Definition of a at the vertex $\boldsymbol{V}$.

As shown in Fig. 3, we define $\boldsymbol{X}_i$, i= 0,...,n, which are the positions of the vertices around the vertex $\boldsymbol{P}_i$ whose new position is $\boldsymbol{P}_i'$ as follows:

$$\boldsymbol{X}_i = (X_{xk}, X_{yk}, X_{zk}) \quad (k = 0, \cdots n) \tag{5.3}$$

$$\boldsymbol{x}_i = \boldsymbol{X}_k - \boldsymbol{P}_i' = \boldsymbol{X}_i - (\boldsymbol{P}_{ic} + \boldsymbol{\phi}\, \boldsymbol{N}_i) \tag{5.4}$$

The angle between $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$ is $\boldsymbol{\theta}_k$ and the area of $\Delta \boldsymbol{P}_i' \boldsymbol{X}_k \boldsymbol{X}_{k+1}$ is s_k. They are expressed by

$$\boldsymbol{\theta}_k = \boldsymbol{cos}^{-1}\left(\frac{\boldsymbol{x}_k \cdot \boldsymbol{x}_{k+1}}{|\boldsymbol{x}_k \cdot \boldsymbol{x}_{k+1}|}\right) \tag{5.5}$$

$$s_k = \frac{1}{2}\left\{|\boldsymbol{x}_k|^2 |\boldsymbol{x}_{k+1}|^2 - (\boldsymbol{x}_k \cdot \boldsymbol{x}_{k+1})^2\right\}^{\frac{1}{2}} \tag{5.6}$$

where $\boldsymbol{x}_k \cdot \boldsymbol{x}_{k+1}$ denotes the scalar product of the vectors $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$. If we use $\boldsymbol{P}_{ic}^k = \boldsymbol{P}_{ic} - \boldsymbol{X}_k$, $\boldsymbol{x}_k \cdot \boldsymbol{x}_{k+1}$ is given by a quadratic function of ϕ as follows:

$$\boldsymbol{x}_k \cdot \boldsymbol{x}_{k+1} = |\boldsymbol{N}_i|^2 \boldsymbol{\phi}^2 + \boldsymbol{N}_i \cdot (\boldsymbol{P}_{ic}^k + \boldsymbol{P}_{ic}^{k+1})\boldsymbol{\phi} + \boldsymbol{P}_{ic}^k \cdot \boldsymbol{P}_{ic}^{k+1} \tag{5.2}$$

5.2 Implementation of the Surface Filter

For implementation of the surface filter, we use the bisection method to determine in Eqn. (5.1). When the distance between the connected vertices in a given meshed surface is sufficiently small, we can assume that the vertex under processing and its neighborhood vertices are on the same plane. If ϕ in Eqn. (5.1) increases, then the Gaussian curvature also increases. Although the Gaussian curvature becomes negative according to the shape of the surface, it is usually possible to determine ϕ by extending the search range. In the case where a suitable ϕ is not found, the vertex $\boldsymbol{P}_i$ is moved to $\boldsymbol{P}_{ic}$.

5.3 Application Examples

We measured a plastic car model (1/24 size) and applied our surface filter to a surface of its hood part as shown in Fig. 3. We used $\alpha = \beta = 1$ for the surface filter. The number of vertices in this surface is 8,205 and that of the triangles is 2,735.

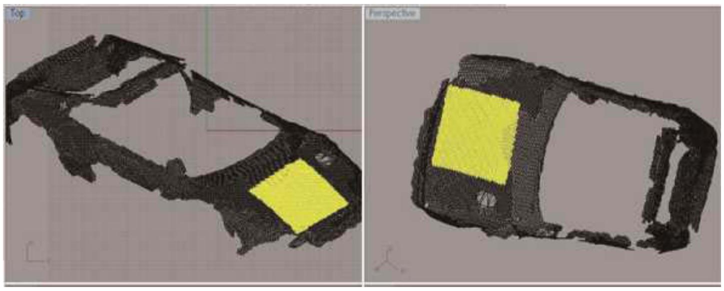


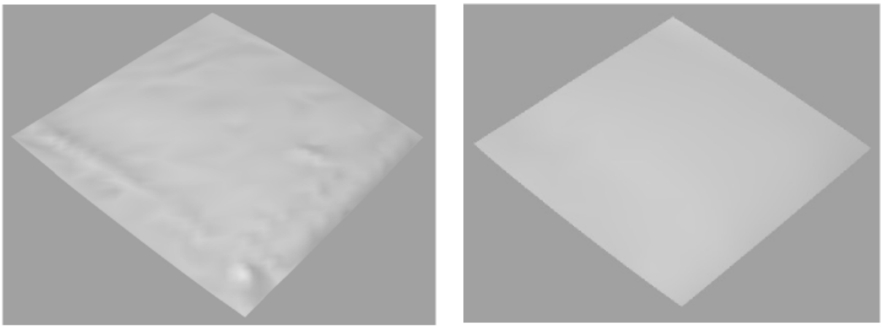
Fig. 3: Measurement data.

Figures 4 to 7 show the rendering images, the distribution of the Gaussian curvature and zebra mapping before and after filtering. It can be seen that the noise was removed and a high- quality surface was obtained. We used a PC with Core i7 2.80 GHz CPU and the processing time was 1.59 s for 10 times filtering.

6 CONCLUSIONS AND FUTURE WORK

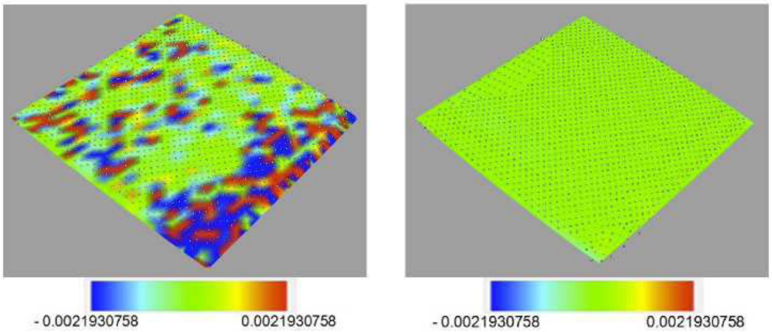
We have proposed a formulation of the log- aesthetic surface by use of the variational principle. We have also implemented a discrete surface filter in the simplest form constructed based on the log-aesthetic surface formulation and found that the filter is effective to remove noise and yields high-quality surfaces by applying it to practical measurement data.

As future work, we will implement optimization codes using Eqn. (4.12) and Eqn. (4.20) to establish the formulation of the log- aesthetic surface.



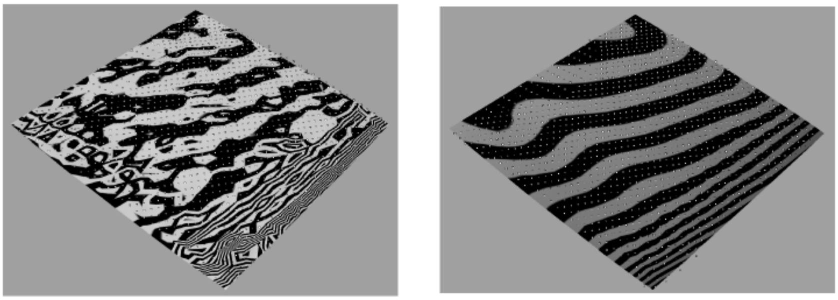
(a) Before filtering (b) After filtering

Fig. 4: Measurement data.



(a) Before filtering (b) After filtering

Fig. 5: Distribution of the Gaussian curvature.



(a) Before filtering (b) After filtering

Fig. 6: Zebra mapping.

ACKNOWLEDGEMENTS

This work was supported in part by the Adaptable and Seamless Technology Transfer Program through target-driven R&D, JST.

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