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A New Method of Fitting Implicit Conic to Plane Scattered Data Points

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ABSTRACT

A new method of fitting implicit conic to plane scattered data points is presented in this paper, which is based on minimizing the sum of squared point-to-curve algebraic distance. At first, the specific ellipse, hyperbola and parabola are fitted to the data points respectively, then the final fitting conic is produced by combining the above three specific conics and adding certain weights to the coefficients of them. By this method, the fitting conic not only preserves the original curve shapes if the conic data comes from the basic quadratic curve, but also improves the fitting effects for the general data points.

Keywords: conic, fitting, type-specific, weighted combination, algebraic distance.
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1 INTRODUCTION

Conic fitting is a basic technology, and plays an important role in the fields of pattern recognition, computer vision, reverse engineering, computer graphics, statistics and computer aided geometric design. In these fields, Conic fitting is especially important for the reverse engineering, whose core is to reconstruct the curve from the sampling points, and request the fitting curve reflecting the shapes and features of the original data points [1-2]. The requirement of reverse engineering leads many researchers to study the shape-preserving conic fitting of the ordered data points.

For the conic fitting, there are two problems to be considered. One is the expression form of the fitting conic, and the other one is the objective function of the fitting conic. In resent years, because of the requirement of expressing complex shape, the implicit algebraic curve and surface have attracted high attention in the field of CAGD for its unique advantages [3-5]. Since the conic has a relatively simple exhibition form, a low degree, good geometric characteristics, and flexible control parameters [6], the research about it becomes wider and the applications of it are in an invincible position. During the fitting of the curve, the selection of the objective function can be divided into two categories primarily: the first category is based on minimizing the point-conic geometric distance and the second one is based on minimizing the algebraic distance. If the objective function is based on minimizing the geometric distance, a 4th order equation is obtained in final, which can be solved in mathematical theory, but it needs to use non-linear method and the results are not stable [7-8]. So the applications of this method are restricted in some level. Therefore, many researches have been done based on minimizing the algebraic distance. Liu used six different constraints to obtain six basic conics, and produced the final fitting conic by adding certain weights to the coefficients of the six basic conics [9]. Rosin discussed the objective functions based on minimizing algebraic distance from the aspects of curvature bias, singularities, and transformational invariance, and also showed the results for the constraints $F = 1$ and $A + C = 1$, respectively [10]. Bookstein provided a generic quadric constraint in matrix form, which is substituted to the variable matrix C [11]. Rosin produced many fitting error functions for the ellipse fitting, and explained the meanings of every function [12]. In paper [13], Fitzgibbon introduced the elliptical constrain to the objective function, transformed it to the extreme problem, and obtained the fitting results by the eigensystem. Harker used the partition matrix to fitting the type-specific conic and introduced a bias correction method [14].

Based on the aforementioned researches, a new conic fitting method is provided in this paper, which is based on minimizing the point-to-curve algebraic distance for the given data points. At first, we fit a specific type conic to the data points, thus finding the best ellipse, hyperbola and parabola respectively, then combining these three curves by adding certain weights to the coefficients of every conic and the weights are solved by minimizing the algebraic distance. This new method can preserve the original shapes of the conic data points, meanwhile, the fitting effects are improved.

The remaining parts of the paper are arranged as follows. The basic idea of the conic fitting is described in Section 2. The methods of fitting ellipse, hyperbola and parabola are discussed in Section 3, respectively. In Section 4, we explain the solving by adding weights to the coefficients of the three specific curves. The experiments are performed in Section 5, and the conclusion is given in Section 6.

2 CONIC FITTING

For the given scattered data points $P_i(x_i, y_i), i = 1 \ldots n$, we will fit a conic based on the minimal algebraic distance. The conic is expressed as follows:

$$Q(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{2.1}$$

So the objective function is to minimize the following formula, thus

$$S = \sum\nolimits_{i=1}^{n} Q(x_i, y_i)^2 = \sum\nolimits_{i=1}^{n} (ax_i^2 + bx_i y_i + cy_i^2 + dx_i + ey_i + f)^2 \tag{2.2}$$

Expressing the above formulas by matrix form, that is

$$Q(x,y) = dz = 0 \tag{2.3}$$

$$S = (Dz)^2 \tag{2.4}$$

where

$$d = [x^2 \quad xy \quad y^2 \quad x \quad y \quad 1], z = [a \quad b \quad c \quad d \quad e \quad f]^T$$

$$D = \begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_n^2 & x_n y_n & y_n^2 & x_n & y_n & 1 \end{bmatrix}$$

Our goal is to get the value of coefficients z by Minimizing Eqn.(2.4). In order to avoid the trivial solution, a constraint is needed. During the coefficients z solution of the fitting conic in this paper, we first fit the type-direct conics of ellipse, hyperbola and parabola, then combining the three obtained coefficients by adding certain weights to them, and regarding the combined results as the coefficients of the final conic.

3 THE TYPE-SPECIFIC FITTING

3.1 The Ellipse Fitting

The constraint for the ellipse fitting is $b^2 - 4ac < 0$. The imposition of this inequality constraint is difficult in general, but we have the freedom to arbitrarily scale the parameters, so we may simply incorporate the scaling into the constraint and impose the equality constraint $b^2 - 4ac = -1$ [13], which may be expressed in the matrix form

$$z^T C z = -1 \tag{3.1}$$

where

$$C = \begin{bmatrix} 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the algorithm for computing the fitting ellipse presented in reference [13] was very typical, which supposed the constraint $b^2 - 4ac = -1$, we also adopt it in this paper. Thus, a Lagrange extreme function for Eqn.(2.4) and Eqn.(3.1) can be constructed as Eqn.(3.2)

$$H(z,\lambda) = z^T D^T D z + \lambda(z^T C z + 1) \tag{3.2}$$

Minimizing Eqn.(3.2) by solving the first derivative for z and λ, we can get

$$D^T D z + l\, C z = 0 \tag{3.3}$$

$$z^T C z = \alpha = -1 \tag{3.4}$$

Eqn.(3.3) and Eqn.(3.4) can be solved as a generalized eigenvector problem, where λ is the eigenvalue of the matrix $(D^T D, C)$, and z is the corresponding eigenvectors. According to the matrix theory, for $sign(\lambda(D^T D, C)) = sign(\lambda(C))$, and $sign(\lambda) = sign(\alpha)$ [13], the matrix C has one negative eigenvalue and two positive eigenvalues. So we can get one ellipse coefficient corresponding to the negative eigenvalue by solving the generalized matrix $(D^T D, C)$.

3.2 The Hyperbola Fitting

Similar to the fitting of ellipse, for the hyperbola fitting, we can choose the constraint $b^2 - 4ac = 1$ [14], thus

$$z^T C z = 1 \tag{3.5}$$

Constructing a Lagrange extreme function for Eqn.(2.4) and Eqn.(3.5), that is

$$H(z,\lambda) = z^T D^T D z + \lambda(z^T C z - 1) \tag{3.6}$$

From Eqn.(3.6), solve the first derivative to z and λ, respectively, and we can get

$$D^T D z + \lambda C z = 0 \tag{3.7}$$

$$z^T C z = \alpha = 1 \tag{3.8}$$

Eqn.(3.7) and Eqn.(3.8) also can be solved by generalized eigenvector, and two positive eigenvalues of the generalized matrix $(D^T D, C)$ are all the coefficients of hyperbola. According to the local extrema characteristics, the lowest eigenvalue just corresponds to the lowest objective function, meanwhile, the fitting hyperbola has least algebraic distance. So we select the eigenvalue with lowest magnitude as the coefficients of the fitting hyperbola.

3.3 The Parabola Fitting

For the parabola fitting, the method we proposed in this paper is simple to understand. We weighted combine the coefficients of the fitting ellipse and hyperbola, and ensuring the discriminant $\Delta = b^2 - 4ac$ equal to be zero. Let z_e, z_h, z_p express the coefficients of ellipse, hyperbola and parabola respectively, z_e, z_h are known, and z_p can be expressed as

$$z_p = (1-\mu) z_e + \mu z_h \tag{3.9}$$

where

$$z_e = \begin{bmatrix} a_e & b_e & c_e & d_e & e_e & f_e \end{bmatrix}^T$$
$$z_h = \begin{bmatrix} a_h & b_h & c_h & d_h & e_h & f_h \end{bmatrix}^T$$
$$z_p = \begin{bmatrix} a_p & b_p & c_p & d_p & e_p & f_p \end{bmatrix}^T$$

Let

$$b_p^2 - 4a_p c_p = 0$$

thus

$$\left[(1-\mu)b_e + \mu b_h\right]^2 - 4\left[(1-\mu)a_e + \mu a_h\right]\left[(1-\mu)c_e + \mu c_h\right] = 0 \tag{3.10}$$

from Eqn.(3.10), we can get

$$\mu = \frac{A - B \pm \sqrt{B^2 - AC}}{A - 2B + C}$$

where

$$A = b_e^2 - 4a_e c_e$$
$$B = b_e b_h - 2a_e c_h - 2a_h c_e$$
$$C = b_h^2 - 4a_h c_h$$

The range of μ is $\mu \in (0,1)$. Because Eqn.(3.10) is a quadratic about μ, the value of Δ is less than zero when $\mu = 0$, and is greater than zero when $\mu = 1$, which means that there is only one value of μ that contents $\Delta = 0$ between 0 and 1. By substituting this μ value to Eqn.(3.9), the coefficients of parabola can be obtained.

4 DETERMINING THE FINAL FITTING CONIC

In this section, we suppose that the coefficients of the ellipse, hyperbola and parabola have been determined by the above mentioned methods, which are expressed as z_e, z_h, z_p. The final fitting conic coefficients are described by their weighted combination and the sum of the weights is 1. The weight values can be solved by minimizing the point-to-curve algebraic distance.

The coefficients of the final conic are described as follows

$$z = \alpha z_e + \beta z_h + (1-\alpha-\beta) z_p \tag{4.1}$$

The sum of the squared algebraic distance of the point-to-curve is

$$S = (Dz)^2 = z^T D^T D z \tag{4.2}$$

Combining Eqn.(4.1) and Eqn.(4.2), we can get

$$S = \alpha^2 r_1 + 2\alpha\beta r_2 + 2(\alpha - \alpha^2 - \alpha\beta) r_3 + \beta^2 r_4 + 2(\beta - \alpha\beta - \beta^2) r_5 + (1-\alpha-\beta)^2 r_6 \tag{4.3}$$

where

$$Q = D^T D$$
$$r_1 = z_e^T Q z_e, r_2 = z_e^T Q z_h, r_3 = z_e^T Q z_p$$
$$r_4 = z_h^T Q z_h, r_5 = z_h^T Q z_p, r_6 = z_p^T Q z_p$$

In order to get the weights, solve the first derivative of α and β in Eqn.(4.3),

$$\frac{\partial S}{\partial \alpha} = \alpha(r_1 - 2r_3 + r_6) + \beta(r_2 - r_3 - r_5 + r_6) + r_3 - r_6 = 0 \tag{4.4}$$

$$\frac{\partial S}{\partial \beta} = \alpha(r_2 - r_3 - r_5 + r_6) + \beta(r_4 - 2r_5 + r_6) + r_5 - r_6 = 0 \tag{4.5}$$

from Eqn.(4.4) and Eqn.(4.5), we can get

$$\alpha = \frac{\begin{vmatrix} r_6 - r_3 & r_2 - r_3 - r_5 + r_6 \\ r_6 - r_5 & r_4 - 2r_5 + r_6 \end{vmatrix}}{\begin{vmatrix} r_1 - 2r_3 + r_6 & r_2 - r_3 - r_5 + r_6 \\ r_2 - r_3 - r_5 + r_6 & r_4 - 2r_5 + r_6 \end{vmatrix}}$$

$$\beta = \frac{\begin{vmatrix} r_1 - 2r_3 + r_6 & r_6 - r_3 \\ r_2 - r_3 - r_5 + r_6 & r_6 - r_5 \end{vmatrix}}{\begin{vmatrix} r_1 - 2r_3 + r_6 & r_2 - r_3 - r_5 + r_6 \\ r_2 - r_3 - r_5 + r_6 & r_4 - 2r_5 + r_6 \end{vmatrix}}$$

By substituting the values of α and β into Eqn.(4.1), the coefficients of the final fitting conic are obtained. Since the computation of α and β is only based on the minimal algebraic distance from the data points to the combined conic, and without any constraint about the discriminant, the result conic can make the defects of the fitting ellipse, hyperbola, parabola balanced. Besides, the conics solved by this new method have the following characteristics, thus

The conic is shape-preserved, which means that if the data points are from certain types of the basic quadratic curve with small noise, thus the ellipse, hyperbola or the parabola, the fitting conic can preserve the origin shape of the data points.

Proof: Without loss of generality, suppose the data points are taken from ellipse precisely, then the fitting conic should be as the same as this ellipse. For the coefficients of the fitting conic $z = \alpha z_e + \beta e_h + (1-\alpha-\beta) z_p$, when $\alpha = 1$ and $\beta = 0$, the distance between the data points and the fitting conic is zero, but using the least squares to solve the values of α, β, only one extrema can be obtained, so we can get the precise ellipse fitting for the data points, when $\alpha = 1$ and $\beta = 0$. Thus the new method can ensure the fitting conic is shape-preserved.

5 EXPERIMENTS

In this section, we list two comparison results, the first one is for the basic quadratic data points, and the other one is for the general data points.

In the following figures, we use the alphabets ***O,R,G,B,P*** to represent the original curves, red curves, green curves, blue curves and the pink curves respectively.

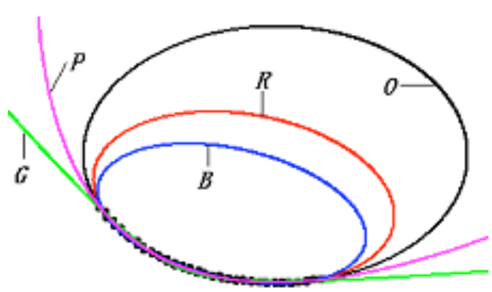

Fig. 1: Elliptical data points.

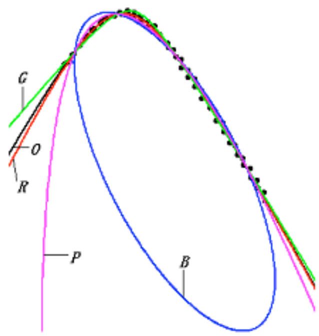

Fig. 2: Hyperbola data points

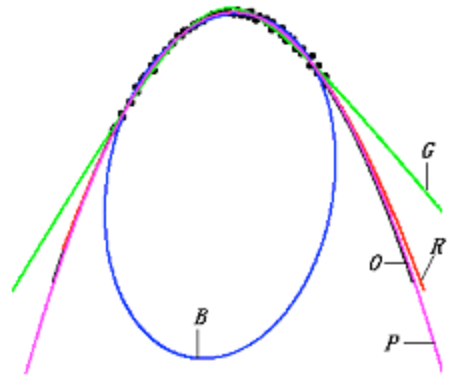

Fig. 3: Parabola data points

Fig. 1-3 illustrate the first comparison results, and the black data points are from ellipse, hyperbola and parabola (the black curves in Fig. 1-3) with small noise respectively. The blue curves, green curves and the pink curves are the fitting curves of ellipse, hyperbola and parabola based on minimal algebraic distance, while the red curves are fitted by the new method. We can see that, for these three data points, the red fitting curves are ellipse, hyperbola and parabola respectively, which preserve the shape of the origin curves, and fit the data points precisely. For example, the red ellipse is closer to the origin ellipse than the blue ellipse in Fig.1.

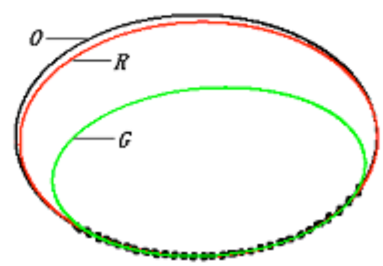

Fig. 4: Ellipse data points

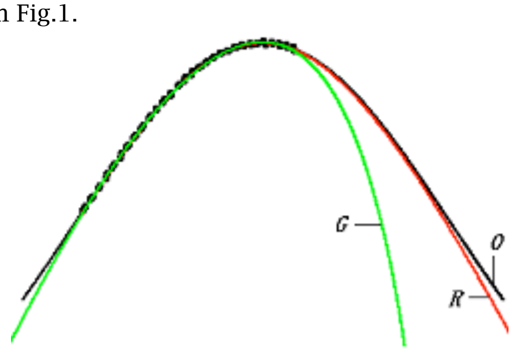

Fig. 5: Cosine curve data points

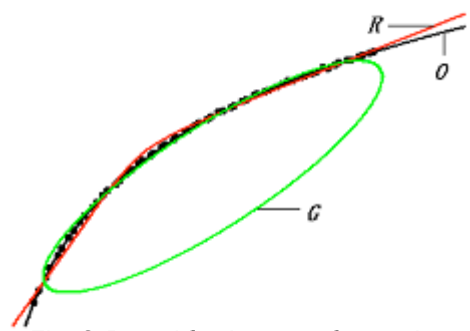

Fig. 6: Logarithmic curve data points

The second comparison results are given in Fig.4-6, where the black data points are from the ellipse, cosine curve and the logarithmic curve (the black curves in Fig.4-6) and with small noise. To fit the conic based on the minimal algebraic distance, the green curves in these three figures are obtained by the old method with constraint $a^2+b^2+c^2+d^2+e^2+f^2=1$, while the red curves are fitted by the new method. The maximum geometric point-to-curve distance (MAX GD) and the sum of the geometric point-to-curve distance (SUM GD) of the above two fitting curves are listed in Tab. 1. It shows that, the fitting results of the new methods are better, especially for Fig. 5, the fitting curve nearly restore the original curve.

Date points	*Old Method*		*New Method*	
	MAX GD	*SUM GD*	*MAX GD*	*SUM GD*

Ellipse	0.187	3.245	0.119	2.634
Cosine curve	0.217	3.403	0.138	2.745
Logarithmic curve	1.514	13.799	0.575	12.387

Tab. 1: The distance comparison

Date points	*Old Method*		*New Method*	
	MAX GD	*SUM GD*	*MAX GD*	*SUM GD*
Descartes curve	0.421	8.428	0.417	8.323
Probability curve	1.999	4.543	1.683	4.153
Versiera	2.569	6.135	1.795	4.065
Cycloid	0.297	5.840	0.258	5.593

Tab. 2: The distance comparison

Besides, we also do the second experiment on the data points that from the other curves. Such as the Descartes curve $x=\frac{3k}{1+k^3}, y=\frac{3k^2}{1+k^3}, k=-0.4$, probability curve $y=e^{-x^2}$, versiera $y=\frac{4a^3}{x^2+4a^2}, a=0.45$ and cycloid $x=\theta-\sin\theta$, $y=1-\cos\theta$. The comparisons of the MAX GD and the SUM GD are shown in Tab. 2, it's also clear that the new method is better than the old method.

6 CONCLUSIONS

A new method of fitting conic to the scattered data points is presented in this paper. At first, we fit the type-specific conic (ellipse, hyperbola and parabola) to the given data points based on minimizing the algebraic distance respectively. Then based on the obtained three conics, the finial curve is constructed by the weighted combination of them. Using the new method to solve the conic fitting problem, especially for the basic quadratic data points, if the curve shapes can't be determined in advance, the results are well and can preserve the origin shapes. Besides, for the mostly general data points, the new method can also improve the fitting effect.

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