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A Computational Geometric Solution of the Kinematic Registration Problem Using the Bisecting Linear Line Complex

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ABSTRACT

This paper presents a computational geometric solution to the kinematic registration problem using computational line geometry. Kinematic registration involves computation of the screw parameters of a motion from specified positions of geometric features of the moving body. The problem is formulated using a special complex of lines associated with kinematics namely the bisecting linear line complex. In this fashion the problem is reduced to an approximation problem in the line space and a slightly modified version of the line approximation method developed by Pottmann, Peternell, and Ravani (1999) is used to find the solution.

Keywords: computational line geometry, registration, linear line complex.
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1. INTRODUCTION

Given two positions of a rigid body in space, there exists a line in space, which the body simultaneously rotates around and translates along. This line which is called the "screw axis" together with the translation distance and rotation angle about it, are called the screw parameters of the motion. Kinematic registration involves the computation of the screw parameters from the homologous geometric features of the body at two positions. Homologous points (lines, …) are points (lines, …) occupied in the fixed space by the repeated positions of the same point (line, …) of the moving body (see [5] p. 35).

Various methods have been developed to solve for the screw parameters. For the case when the homologous points are known, Beggs [4] obtained a solution by solving a set of algebraic equations. Laub and Shiflett [13] used linear algebra and matrix perturbation theory. Angeles [1-2] solved the problem based on the invariant concept of a second-order tensor. His solution overcomes the lack of symmetry in point feature ordering and singularities found in [13]. A detailed comparison between these and other methods can be found in [9]. Earlier studies had only used homologous point features of the body. Combinations of line, plane, and point features of the body were first considered by Ravani and Ge [17] to solve the kinematic registration problem. They also formulated an over-determined system and reduced the problem to the solution of a linear system of equations [10].
Eberharter and Ravani [7] developed a geometric solution to the problem by utilizing Reuleaux's 2D method [18]. They found a special plane in space and constructed the 2D solution on the plane. Recently, Baroon and Ravani [3] constructed a geometrical 3D generalization of Reuleaux's method based on line geometry. They have also found the optimal screw parameters for the over-determined case. A modern and interesting application of such problems in CAD and visualization can be found in Rossignac and Kim [19]. In this paper we present a solution of the kinematic registration problem

using computational line geometry. The results show another application of the evolving field of computational line geometry. For a detailed exposition to computational line geometry the readers are referred to Pottmann and Wallner [16].

2. THE LINEAR LINE COMPLEX

The Euclidean space, E^3, contains ∞^4 lines. A linear line complex is the set of ∞^3 lines $L=(l,\bar{l})$ in their Plücker coordinates (see Appendix A.1) that satisfies the linear relationship [16]

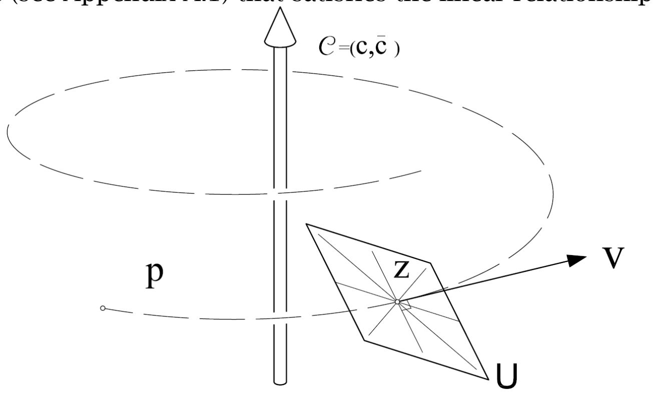

Fig. 1: Helical motion of a point and its null plane.

$$c\cdot\bar{l}+\bar{c}\cdot l=0 \tag{2.1}$$

The sextuple vector $C=(c,\bar{c})$ is called the axis of the complex. The above equation has the same form as the reciprocal product between two lines or between a line and a screw. Therefore, every line that belongs to the linear line complex is reciprocal to the *C* axis. There are two distinct types of linear line complexes: (a) if $c\cdot\bar{c}=0$, i.e., *C* constitutes the Plücker coordinates of a line, then the linear line complex is called singular; (b) if $c\cdot\bar{c}\neq 0$, then the linear line complex is called regular.

2.1 The Linear Line Complex in Kinematics

In this section, we will show some of the relationships between the linear line complex and kinematics. An extensive discussion on this subject can be found in [7-8],[14],[16].

Let the axis $C=(\mathbf{c},\bar{\mathbf{c}})$ be the screw coordinates of the continuous helical motion of a point *p* as shown in Fig. 1. The helical motion of a point **z** is given by [5]

$$z=\mathbf{A}p+r \tag{2.2}$$

where **A** is an orthogonal matrix and *r* is a 3D vector. The velocity of point *p* is then expressed as

$$v=\dot{z}=\dot{\mathbf{A}}p+\dot{r}=\dot{\mathbf{A}}\mathbf{A}^Tz-\dot{\mathbf{A}}\mathbf{A}^Tr+\dot{r}=c\times z+\bar{c} \tag{2.3}$$

The path normals are the lines that pass through the point **z** and perpendicular to the velocity vector **v** as shown in Fig. 1. Let the Plücker coordinates of the path normals be $L=(l,\bar{l})$. Path normals form a pencil of lines, and each path normal satisfies the equation

$$0=v\cdot l=(c\times z)\cdot l+\bar{c}\cdot l=c\cdot\bar{l}+\bar{c}\cdot l \tag{2.4}$$

The above equation proves that the path normals to the trajectory of points under a helical motion form a linear line complex with an axis that is identical to the screw axis of the helical motion.

The pencil lies in a plane called the path normal plane or null plane [15]. The center of the null plane where all of the path normals intersect is called the null point (see Fig. 1). If the null point is represented in its homogeneous coordinates as $Z=(z_0,z)$, then the homogeneous plane coordinates of the null plane in terms of the linear line complex axis is given by

$$U=(u_0,u)=(z\cdot\bar{c},-z_0\bar{c}+z\times c) \tag{2.5}$$

Conversely, the null point's homogeneous coordinates can be expressed in terms of the null plane's homogeneous plane coordinates as

$$Z = (z_0, z) = (u \cdot c, -u_0 c + u \times \bar{c}) \tag{2.6}$$

2.2 The Bisecting Linear Line Complex

The bisecting linear line complex is a special form of a linear line complex which can be used to describe the computational geometry of two position theory in kinematics. Some of its properties have been described in [6],[11].

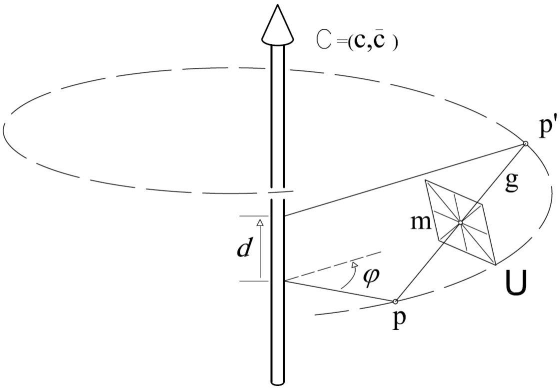

Fig. 2: The bisecting plane between two positions of a point.

Consider two positions of a rigid body before and after a displacement. The screw parameters are the screw axis $C = (c, \bar{c})$, the rotation angle ϕ and translation distance *d*. The combination of the rotation and the translational movements is the screw or helical motion of the body.

Let p be the point denoting the first position and p' be its corresponding or homologous point after displacement as shown in Fig. 2. The line $g_i = p_i' - p_i$ connects the two points, and the point $m_i = (p_i' + p_i)/2$ is their midpoint. All lines that are perpendicular to the connecting line g_i and pass though the midpoint m_i form a pencil of lines embedded in the bisecting plane *U*.
Defining the pitch of a linear line complex as

$$P = \frac{h/2}{\tan \phi / 2} \tag{2.7}$$

Eberharter [6] proved that the union of the bisecting line pencils over all midpoint homologous pairs belongs to a linear line complex. Therefore, every line $L = (l, \bar{l})$ in a bisecting pencil satisfies the linear reciprocal equation

$$\Omega(L, C) = l \cdot \bar{c} + \bar{l} \cdot c \tag{2.8}$$

Therefore, for two positions of a rigid body in space, there exists a bisecting linear line complex that is formed by the union of the bisecting pencil of lines that is perpendicular to the connecting line and is incident with the midpoint. This bisecting linear line complex has the same screw axis of the helical motion of the body from the first position to the second position. Its pitch is defined by equation (2.7). When the positions of the two points are infinitesimally close, the connecting line between the two points will become the direction of the velocity vector of the motion. Thus, the bisecting plane is the null plane of the motion. In the infinitesimal case, the pitch of the linear line complex will reduce to

$$P = \lim_{\phi \to 0} = \frac{h/2}{\tan \phi / 2} = \frac{h}{\phi} \tag{2.9}$$

3. LINEAR LINE COMPLEX FITTING

Pottmann, Peternell, and Ravani (1999) [15] developed an algorithm to fit a linear line complex from a given set of *k* lines (see Appendix A.2). In this section, we will develop an algorithm to fit an exact linear line complex from a set of three null planes. A null plane can contain the path normals, which form a pencil of lines, of the trajectory of a point undergoing a helical motion, or it can contain the midpoint bisecting pencil of lines between two positions of a rigid body. Every line in such pencils of lines belongs to the linear line complex. We will also develop an algorithm for the over-determined case of fitting from *n* null planes.

3.1 Exact Fitting from Three Null Planes

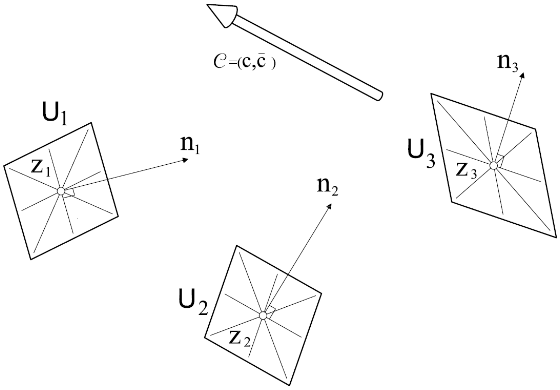

Fig. 3: Null planes.

A minimum of three null planes is required to fully specify a linear line complex. For a linear line complex with the axis $C = (c, \bar{c})$, the homogeneous plane coordinates of the three null planes are expressed in terms of the linear line complex axis and the null points $Z_i = (1, z_i)$ as in [15]:

$$\begin{aligned} U_1 &= (z_1 \cdot \bar{c}, -\bar{c} + z_1 \times c) \\ U_2 &= (z_2 \cdot \bar{c}, -\bar{c} + z_2 \times c) \\ U_3 &= (z_3 \cdot \bar{c}, -\bar{c} + z_3 \times c) \end{aligned} \tag{3.1}$$

The same null planes, shown in Fig. 3, can also be represented in terms of the null points and the normal vectors n_i to the planes as

$$\begin{aligned} P_2 &= (-z_1 \cdot n_1, n_1) \\ P_2 &= (-z_2 \cdot n_2, n_2) \\ P_3 &= (-z_3 \cdot n_3, n_3) \end{aligned} \tag{3.2}$$

The normal vector could be the velocity vector of the null point under a helical motion or the connecting line between two homologous points in finite kinematics. Matching equations (3.1) and (3.2) yields two systems of three equations. These systems are solved for the linear line complex axis. The first system is expressed as

$$\begin{aligned} z_1 \cdot \bar{c} &= -z_1 \cdot n_1 \\ z_2 \cdot \bar{c} &= -z_2 \cdot n_2 \\ z_3 \cdot \bar{c} &= -z_3 \cdot n_3 \end{aligned} \tag{3.3}$$

In a matrix form, the above system will be

$$\begin{bmatrix} \cdot & z_1 & \cdot \\ \cdot & z_2 & \cdot \\ \cdot & z_3 & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \bar{c} \\ \cdot \end{bmatrix} = - \begin{bmatrix} z_1 \cdot n_1 \\ z_2 \cdot n_2 \\ z_3 \cdot n_3 \end{bmatrix} \tag{3.4}$$

The solution of $\bar{c}$ will be

$$\begin{bmatrix} \cdot \\ \bar{c} \\ \cdot \end{bmatrix} = - \begin{bmatrix} \cdot & z_1 & \cdot \\ \cdot & z_2 & \cdot \\ \cdot & z_3 & \cdot \end{bmatrix}^{-1} \begin{bmatrix} z_1 \cdot n_1 \\ z_2 \cdot n_2 \\ z_3 \cdot n_3 \end{bmatrix} \tag{3.5}$$

The moment $\overline{c}$ of the linear line complex is used to find the direction of the axis. The second system of equations that is formed from equations (3.2) and (3.3) is written as

$$\begin{aligned} z_1 \times c &= n_1 + \overline{c} \\ z_2 \times c &= n_2 + \overline{c} \\ z_3 \times c &= n_3 + \overline{c} \end{aligned} \tag{3.6}$$

We multiply each of the above equations by n_1, n_2, and n_3, respectively, and write the results in a matrix form as follows

$$\begin{bmatrix} \cdot & n_1 \times z_1 & \cdot \\ \cdot & n_2 \times z_2 & \cdot \\ \cdot & n_3 \times z_3 & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ c \\ \cdot \end{bmatrix} = \begin{bmatrix} (n_1 + \overline{c}) \cdot n_1 \\ (n_2 + \overline{c}) \cdot n_2 \\ (n_3 + \overline{c}) \cdot n_3 \end{bmatrix} \tag{3.7}$$

The above system is solved for the direction part **c** of the linear line complex as

$$\begin{bmatrix} \cdot \\ c \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot & n_1 \times z_1 & \cdot \\ \cdot & n_2 \times z_2 & \cdot \\ \cdot & n_3 \times z_3 & \cdot \end{bmatrix}^{-1} \begin{bmatrix} (n_1 + \overline{c}) \cdot n_1 \\ (n_2 + \overline{c}) \cdot n_2 \\ (n_3 + \overline{c}) \cdot n_3 \end{bmatrix} \tag{3.8}$$

where $\overline{c}$ is found from equation (3.5). Note that the linear line complex axis $C = (c, \overline{c})$ is in a non-normalized form. It can be normalized as

$$C = \left(\frac{c}{\|c\|}, \frac{\overline{c}}{\|c\|} \right) \tag{3.9}$$

Moreover, the normal vectors n_i should not be normalized prior to forming the matrices.

3.2 Fitting a Linear Line Complex from n Null Planes

In the preceding section, we formulated a solution to find a linear line complex from a set of three planes. In this section, we will develop an algorithm for fitting a linear line complex from more than three planes. In practice, more than three points are generally known, and some of them have error measurements and some uncertainties. This is why it is important to fit all of the given data to the best linear line complex.

The homogeneous coordinates of null planes in terms of the homogeneous coordinates of the null points that are not at infinity, $Z_i = (1, z_i)$, and a linear line complex axis, $C = (c, \overline{c})$, are written as [15]

$$U_i = (z_i \cdot \overline{c}, -\overline{c} + z_i \times c) \qquad i = 1, 2, \ldots, n \tag{3.10}$$

The same null planes are written in terms of the null points and normal vectors to the planes as

$$P_i = (-z_i \cdot n_i, n_i) \qquad i = 1, 2, \ldots, n \tag{3.11}$$

Equating equations (3.10) and (3.11) with some algebraic modifications yields

$$z_i \cdot \overline{c} + z_i \cdot n_i = 0 \qquad i = 1, 2, \ldots, n \tag{3.12}$$

$$(z_i \times n_i) \cdot c + n_i \cdot (n_i + \overline{c}) = 0 \qquad i = 1, 2, \ldots, n \tag{3.13}$$

We will utilize the least squares method to solve equation (3.12) for $\overline{c}$. We will then use the result to solve for **c** in equation (3.13). The sum of square errors of equation (3.12) is written as

$$E_{\overline{c}} = \sum_{i=1}^{n} (z_i \cdot \overline{c} + z_i \cdot n_i)^2 \tag{3.14}$$

The above equation is equivalent to

$$E_{\overline{c}} = \overline{c}^T \mathbf{Z}^T \mathbf{Z} \overline{c} + 2\mathbf{W}^T \mathbf{Z} \overline{c} + \mathbf{W}^T \mathbf{W} \tag{3.15}$$

where **Z** is an $n \times 3$ matrix formed from the null points as its rows. The matrix **Z** is written as

$$\mathbf{Z} = \begin{bmatrix} \cdot\, z_1 \,\cdot \\ \cdot\, z_2 \,\cdot \\ \cdots \\ \cdot\, z_n \,\cdot \end{bmatrix} \tag{3.16}$$

and the matrix **W** is an $n \times 1$ column vector formed as

$$\mathbf{W} = \begin{bmatrix} z_1 \cdot n_1 \\ z_2 \cdot n_2 \\ \cdots \\ z_n \cdot n_n \end{bmatrix} \tag{3.17}$$

Minimizing equation (3.15) and solving for $\overline{c}$ yields

$$\overline{c} = -(\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{Z}^T\mathbf{W} \tag{3.18}$$

The result of $\overline{c}$ is used to solve for the direction vector c of the linear line complex. Using the least squares approach, the sum of square errors in equation (3.13) is

$$E_c = \sum_{i=1}^{n}\left[(z_i \times n_i)\cdot c + n_i \cdot (n_i + \overline{c})\right]^2 \tag{3.19}$$

The above equation can be expressed as

$$E_c = c^T\mathbf{E}^T\mathbf{E}c + 2\mathbf{F}^T\mathbf{E}c^T + \mathbf{F}^T\mathbf{F} \tag{3.20}$$

where E is an $n \times 3$ matrix written as

$$\mathbf{E} = \begin{bmatrix} \cdot\, z_1 \times n_1 \,\cdot \\ \cdot\, z_2 \times n_2 \,\cdot \\ \cdots \\ \cdot\, z_n \times n_n \,\cdot \end{bmatrix} \tag{3.21}$$

and the matrix F is an $n \times 1$ column vector formed as

$$\mathbf{F} = \begin{bmatrix} n_1 \cdot (n_1 + \overline{c}) \\ n_2 \cdot (n_2 + \overline{c}) \\ \cdot \\ n_n \cdot (n_n + \overline{c}) \end{bmatrix} \tag{3.22}$$

We minimize equation (3.20) and solve for the direction **c** of the linear line complex axis

$$c = -(\mathbf{E}^T\mathbf{E})^{-1}\mathbf{E}^T\mathbf{F} \tag{3.23}$$

Note that neither the normal vectors of the null planes nor the linear line complex axis are normalized. The normalized form of the linear line complex will be

$$C = \left(\frac{c}{\|c\|}, \frac{\overline{c}}{\|c\|}\right) \tag{3.24}$$

The pitch of the linear line complex is found from

$$P = \frac{c \cdot \overline{c}}{\|c\|^2} \tag{3.25}$$

This algorithm will fit a linear line complex when the normal vectors represent the velocity vectors of the null points or when they represent the connecting lines between homologous points. In the latter case, the result will correspond to the axis of a bisecting linear line complex.

4. KINEMATIC REGISTRATION

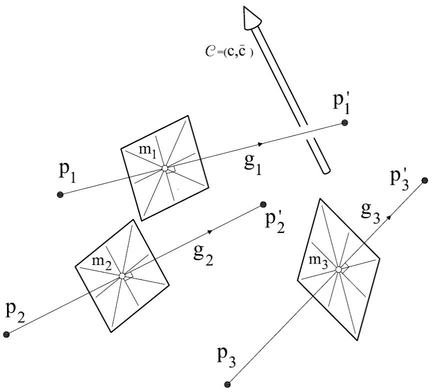

Fig. 4: Homologous points and their bisecting planes.

The linear line complex fitting algorithm from a set of null planes derived in the preceding section will be used to find the screw axis of a helical motion of a rigid body between two different positions. Three pairs of homologous points are sufficient to determine the linear line complex associated with the two distinct positions. This linear line complex has the same axis as the screw motion of the body moving from the first position to the second position. We will also solve the over-determined registration problem where more than three pairs of points are known. This is an important problem because real data points have some measurement errors and uncertainties. The over-determined fitting will allow us to find the best linear line complex of the two positions and the best screw axis.

4.1 Kinematic Registration Based on Three Homologous Points

Let p_1, p_2, and p_3 be three points on the rigid body before displacement; p_1', p_2', and p_3' are their corresponding (homologous) points after displacement as shown in Fig. 4. Points m_1, m_2, and m_3 are the midpoints between the pairs. A line g_i is the connecting line between each point and its homologous one as explained is section 2.2. Every midpoint of a pair with its connecting line forms a null plane of the bisecting linear line complex. All lines that are embedded in the null planes and coincide with the midpoints belong to the bisecting linear line complex. If a bisecting linear line complex axis is written as $C=(c,\bar{c})$, then $\bar{c}$ is found from equation (3.5) as

$$\begin{bmatrix} \cdot \\ \bar{c} \\ \cdot \end{bmatrix} = -\begin{bmatrix} \cdot\, m_1 \,\cdot \\ \cdot\, m_2 \,\cdot \\ \cdot\, m_3 \,\cdot \end{bmatrix}^{-1} \begin{bmatrix} m_1 \cdot g_1 \\ m_2 \cdot g_2 \\ m_3 \cdot g_3 \end{bmatrix} \tag{4.1}$$

The direction vector **c** of the bisecting linear line complex can now be found from equation (3.8) as

$$\begin{bmatrix} \cdot \\ c \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot\ g_1 \times m_1 \ \cdot \\ \cdot\ g_2 \times m_2 \ \cdot \\ \cdot\ g_3 \times m_3 \ \cdot \end{bmatrix}^{-1} \begin{bmatrix} (g_1 + \bar{c}) \cdot g_1 \\ (g_2 + \bar{c}) \cdot g_2 \\ (g_3 + \bar{c}) \cdot g_3 \end{bmatrix} \tag{4.2}$$

The pitch of the bisecting linear line complex axis is written as

$$P_{bllc} = \frac{c \cdot \bar{c}}{c^2} \tag{4.3}$$

The line part of the screw axis of the helical motion between the two positions is the line of the bisecting linear line complex axis. The Plücker coordinates of that line can be found as

$$S = (s, \bar{s}) = (c\cdot, \bar{c} - P_{bllc} c) \tag{4.4}$$

The pitch of the bisecting linear line complex can also be represented in the form [6]

$$P_{bllc} = \frac{d}{2\tan\frac{\phi}{2}} \tag{4.5}$$

where *d* is the translation distance along the axis and ϕ is the angle of rotation around the axis. The translation distance can be easily found from the projection of any of the connecting lines onto the screw axis direction, i.e.,

$$d = (p_i' - p_i)\cdot\frac{s}{\|s\|} \tag{4.6}$$

Then, the rotation angle can be found from

$$\phi = 2\tan^{-1}\left(\frac{d}{P_{bllc}}\right) \tag{4.7}$$

The pitch of the screw axis will be

$$P = \frac{d}{\phi} \tag{4.8}$$

4.2 Kinematic Registration Based on n Homologous Points

The over-determined version of the registration problem can be solved using the linear line complex fitting approach described in section 3.2.2. For a given set of *n* homologous point pairs before and after displacement, we compute *n* midpoints

$$m_i = \frac{(p_i + p_i')}{2} \tag{4.9}$$

and *n* connecting lines $g_i = p_i' - p_i$. Then the vector $\overline{c}$ of the bisecting linear line complex is found from equation (3.18) as

$$\overline{c} = -(\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{Z}^T\mathbf{W} \tag{4.10}$$

where **Z** is an $n \times 3$ matrix

$$\mathbf{Z} = \begin{bmatrix} \cdot\, m_1 \,\cdot \\ \cdot\, m_2 \,\cdot \\ \cdots \\ \cdot\, m_n \,\cdot \end{bmatrix} \tag{4.11}$$

and **W** is an $n \times 1$ matrix

$$\mathbf{W} = \begin{bmatrix} m_1 \cdot g_1 \\ m_2 \cdot g_2 \\ \cdot \\ m_n \cdot g_n \end{bmatrix} \tag{4.12}$$

The direction part *c* of the bisecting linear line complex can now be computed from equation (3.23)

$$c = -(\mathbf{E}^T\mathbf{E})^{-1}\mathbf{E}^T\mathbf{F} \tag{4.13}$$

where E is an $n \times 3$ matrix

$$\mathbf{E} = \begin{bmatrix} \cdot\, m_1 \times g_1 \,\cdot \\ \cdot\, m_2 \times g_2 \,\cdot \\ \cdots \\ \cdot\, m_n \times g_n \,\cdot \end{bmatrix} \tag{4.14}$$

and **F** is the matrix

$$\mathbf{F} = \begin{bmatrix} (g_1 + \overline{c}) \cdot g_1 \\ (g_2 + \overline{c}) \cdot g_2 \\ \cdot \\ (g_n + \overline{c}) \cdot g_n \end{bmatrix} \tag{4.15}$$

The axis $C = (c, \overline{c})$ represents the bisecting linear line complex axis of the over-determined problem. Similar to the exact approach, the screw parameters of the helical motion of the body between the two positions are found from equations (4.3-4.8).

4.3 The Instantaneous Case

The exact and over-determined fitting algorithms developed earlier can be used to solve for the instantaneous screw axis of the instantaneous motion of the body. In the instantaneous case, instead of substituting the midpoints and connecting lines in equations (4.1), (4.2), and (4.10-4.15), we substitute the given point coordinates and their velocity vectors. The results will be the axis of the linear line complex. The linear line complex axis and pitch are the same as the instantaneous screw axis of the motion and its pitch.

4.4 Examples

We solve numerical examples of exact and over-determined registration problems. The screw axis parameters of the helical motion will be extracted from the linear line complex axis.

Example 1

This example is to illustrate how to find the screw axis between two positions given three pairs of homologous points using the linear line complex. The example is taken from Beggs [4]. The three points of the rigid body before displacement that have the coordinates

$$p_1 = (1,1,7) \qquad p_2 = (4,7,1) \qquad p_3 = (7,10,10)$$

Their corresponding coordinates after displacement are

$$p_1' = (7,1,7) \qquad p_2' = (3,9,6) \qquad p_3' = (10,10,3)$$

The midpoints and connecting lines are calculated for each pair of points, and then $\overline{c}$ is computed from equation (4.1) as

$$\overline{c} = (-2,-2,-2)$$

Then, the direction **c** is calculated from equation (4.2):

$$c = \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$$

The axis $C = (c, \overline{c})$ is the bisecting linear line complex axis. It has the same axis as the helical motion but a different pitch. The pitch of the bisecting linear line complex axis is

$$P_{bllc} = \frac{c \cdot \overline{c}}{c \cdot c} = 3$$

Now, the axis of the helical screw is computed from equation (4.4):

$$S = (s, \overline{s}) = (c, \overline{c} - p_{bllc} c)$$
$$= (-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}; 0, 0, 0)$$

The Plücker coordinates of the normalized screw axis will then be

$$S = (\frac{s}{\|s\|}, \frac{\overline{s}}{\|s\|})$$
$$= (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}; 0, 0, 0)$$

and agree with Beggs' results. The trabeation distance is calculated from equation (4.5) as

$$d = (p_i' - p_i) \cdot \frac{s}{\|s\|} = -3.4641016$$

and the rotation angle of the body around the screw axis is found from equation (4.6):

$$\phi = 2\tan^{-1}\left(\frac{h}{p_{bllc}}\right) = -60^o$$

Example 2

This over-determined example is taken from Eberharter and Ravani [7]. Assume that points of the rigid body before displacement have the following coordinates

$$p_1 = (5,0,0) \qquad p_2 = (5,5,0) \qquad p_3 = (0,5,0) \qquad p_4 = (-3,5,0) \qquad p_5 = (-3,-3,-1)$$

and their homologous coordinates are

$$p_1' = (6.238070, 3.413720, 3.976300)$$
$$p_2' = (4.449295, 7.842681, 5.454336)$$
$$p_3' = (-0.056535, 5.790350, 6.151056)$$
$$p_4' = (-2.760034, 4.558951, 6.569088)$$
$$p_5' = (-0.142761, -2.310845, 3.259138)$$

We compute the connecting line g_i and midpoint m_i of every pair. We then form the matrices **Z** and **W** from equations (4.11) and (4.12). The vector $\bar{c}$ is computed from equation (4.10) as

$$\bar{c} = (-1.531516, -1.646094, -4.664481)$$

We use $\bar{c}$ to compute the matrices E and F from equations (4.14) and (4.15). The direction **c** is calculated from equation (4.13) as

$$c = (-0.274459, -0.205845, -0.411689)$$

Similar to the previous example, we compute the pitch of the bisecting linear line complex and the screw axis' Plücker coordinates. The normalized Plücker coordinates of the screw axis are found to be

$$S = (-0.512147, -0.384111, -0.768222; 1.920546, 0.512147, -1.536435)$$

The screw axis is the same axis found in [7]. The translation distance and the rotation angle are computed using equations (4.5) and (4.6), respectively, and they are found to be $h = -5$ and $\phi = -30$.

Example 3

The instantaneous screw axis of a rigid body can be found using the same fitting algorithms derived earlier in section 3.1. This example, from Angeles [2], has the following point coordinates:

$$p_1 = (1,1,7) \qquad p_2 = (4,7,1) \qquad p_1 = (7,10,10)$$

The velocity of each point is:

$$v_1 = (7,-5,1) \qquad v_2 = (-5,4,4) \qquad v_1 = (1,-2,4)$$

The vector $\bar{c}$ will be found from equation (4.1) as

$$\begin{bmatrix} \cdot \\ \bar{c} \\ \cdot \end{bmatrix} = -\begin{bmatrix} \cdot\, p_1 \,\cdot \\ \cdot\, p_2 \,\cdot \\ \cdot\, p_3 \,\cdot \end{bmatrix}^{-1} \begin{bmatrix} p_1 \cdot v_1 \\ p_2 \cdot v_2 \\ p_3 \cdot v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

and the direction vector **c** is computed from equation (4.2) as

$$\begin{bmatrix} \cdot \\ c \\ \cdot \end{bmatrix} = -\begin{bmatrix} \cdot\, v_1 \times p_1 \,\cdot \\ \cdot\, v_2 \times p_2 \,\cdot \\ \cdot\, v_3 \times p_3 \,\cdot \end{bmatrix}^{-1} \begin{bmatrix} (v_1 + \bar{c}) \cdot v_1 \\ (v_2 + \bar{c}) \cdot v_2 \\ (v_3 + \bar{c}) \cdot v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

The pitch of the linear line complex is the same as the pitch of the instantaneous screw motion and it is equal to

$$p = \frac{c \cdot \overline{c}}{c \cdot c} = 1$$

The Plücker coordinates of the screw axis can be found to be

$$S = (s, \overline{s}) = (c, \overline{c} - pc) = (-1, -1, -1; 0, 0, 0)$$

The translation velocity of the body will be the projection of any velocity vector on the normalized direction of the screw

$$v = v_1 \cdot \frac{s}{\left\| s \right\|} = -\sqrt{3}$$

The angular velocity is calculated from the pitch as

$$w = \frac{v}{p} = -\sqrt{3}$$

These results agree with Angeles' computations.

5. CONCLUSIONS

We have developed an algorithm to fit a linear line complex from the data of null planes. We used this algorithm to solve the kinematic registration problem from three homologous pairs of points and the over-determined version of the problem. We also used the same algorithm to find the instantaneous screw parameters of a helical motion of a rigid body. The results represent another application of computational line geometry and are of theoretical interest in evolving the classical field of line geometry into a modern computational framework.

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Appendix

A.1 Plücker Coordinates

One way to represent a line in 3D Euclidean space *E*3 is to use the Plücker coordinates. A line in normalized Plücker coordinates is written as

$$L = (l, \overline{l}) \tag{A.1}$$

where $l = (l_1, l_2, l_3)$ is the direction part of the line under the normalized condition

$$\left\| l \right\| = 1 \tag{A.2}$$

If the point **p** is any point on the line, then the moment part with respect to the origin $\overline{l} = (l_4, l_5, l_6)$ is expressed as

$$\overline{l} = p \times l \tag{A.3}$$

Since l and $\overline{l}$ are orthogonal, a line in Plücker coordinates satisfies what is called the Plücker identity

$$l \cdot \overline{l} = 0 \tag{A.4}$$

A.2 Fitting a Linear Line Complex from Lines

Any five lines define a complex [14]. For the case of more than five lines, Pottmann, Peternell, and Ravani [15] developed an algorithm to fit the lines into a linear line complex. The following is a summary of the procedure. For more details, the reader can refer to [15-16].

The objective of the algorithm is to find the closest linear line complex from a set of *k* lines in their Plücker coordinates. According to Klein [12], the moment of a line L_i with respect to *C* is written as

$$m(L_i, C) = \frac{\left| \overline{c} \cdot l_i + c \cdot \overline{l}_i \right|}{\left\| c \right\|} \tag{A.5}$$

If the moment of the line with respect to the linear line complex axis vanishes, then the line belongs to a linear line complex. The linear line complex that best represents the given set of lines is found from the minimization of

$$\sum_{i=1}^{k} m(L_i, C)^2 \tag{A.6}$$

The above equation is equivalent to minimizing the positive semi-quadric from

$$F(C) = \sum_{i=1}^{k} (\overline{c} \cdot l_i + c \cdot \overline{l}_i) = C^T M C \tag{A.7}$$

where **M** is a 6×6 matrix of the form

$$\mathbf{M} = \sum_{i=1}^{k} \left(\frac{\overline{l}_i \cdot \overline{l}_i^T \left| \overline{l}_i \cdot l_i^T \right.}{l_i \cdot \overline{l}_i^T \left| l_i \cdot l_i^T \right.} \right) \tag{A.8}$$

under the normalization condition

$$1 = \|c\|^2 = C^T \mathbf{D} C \tag{A.9}$$

and **D** is a 6×6 diagonal matrix **D** = $diag(1,1,1,0,0,0)$. The problem is now reduced to solving the system $C^T \mathbf{M} C$ under the constraint: $C^T \mathbf{D} C = 1$. Using the Lagrangian multiplier λ, the Lagrangian function to be minimized is written as

$$E(C) = C^T \mathbf{M} C - \lambda(C^T \mathbf{D} C - 1) \tag{A.10}$$

Minimizing the above function yields

$$(\mathbf{M} - \lambda \mathbf{D})C = 0 \tag{A.11}$$

Pottmann, Peternell, and Ravani [15] showed that λ is the solution of the cubic equation

$$\det(\mathbf{M} - \lambda \mathbf{D}) = 0 \tag{A.12}$$

The solution of the system is an eigenvector *C* that corresponds to the eigenvalue λ. For any root λ, we have

$$F(C) = C^T \mathbf{M} C = \lambda C^T \mathbf{D} C = \lambda \tag{A.13}$$

This means that all of the values of λ's are nonnegative. Therefore, the best solution is the one associated with the smallest λ. If $\lambda = 0$, then the given set of lines forms an exact linear line complex. The standard deviation of the lines from *C* is given by

$$\sigma = \sqrt{\frac{\lambda}{k-5}} \tag{A.14}$$

It is a measure of how close the resulting linear line complex approximates the given lines. The pitch of the linear line complex is computed from

$$p = \frac{c \cdot \overline{c}}{c^2} \tag{A.15}$$