Computer-Aided Design and Applications

http://www.cadanda.com

Interpretable Interval Constraint Solvers in Semantic Tolerance Analysis

Yan Wang

University of Central Florida, wangyan@mail.ucf.edu

ABSTRACT

A semantic tolerance modeling scheme based on generalized intervals was recently proposed to allow for embedding more tolerancing intents in specifications with a combination of numerical intervals and logical quantifiers. By differentiating a priori and a posteriori tolerances, the logic relationships among variables can be interpreted, which is useful to verify completeness and soundness of numerical estimations in tolerance analysis. In this paper, we present a semantic tolerance analysis approach to estimate tolerance stack-ups. New interpretable linear and nonlinear constraint solvers are developed to ensure interpretability of variation estimations. This new approach enhances traditional numerical analysis methods by preserving logical information during computation such that more semantics can be derived from numerical results.

Keywords: tolerance analysis, generalized interval, quantified interval constraint satisfaction.
DOI: 10.3722/cadaps.2008.654-666

1. INTRODUCTION

In tolerance analysis, estimations of accumulative tolerances are mathematically formulated and solved in different ways. The typical approaches include variational estimation, kinematic formulation, statistical approximation, and Monte Carlo simulation. The analysis process is simplified to the computation of pure numerical intervals. Methods of linearization and high-order Taylor approximations are extensively used to compute parameters (e.g., statistical moments) and variables (e.g., kinematic variations in assemblies). Because of these numerical treatments, completeness and soundness of range estimations are compromised. A *complete* solution includes all possible occurrences, which is to check if the range estimation includes all possible stack-up results. Conversely, a *sound* solution does not include impossible occurrences, which consists in checking if the interval overestimates the actual range.

The traditional worst-case linear stack-up methods focus on completeness while range estimations may not be sound. The results usually are overly pessimistic. In contrast, Monte Carlo methods focus on soundness while estimations may not be complete. Assuming the applied distributions and their parameters reflect the true variations, the simulated ranges are complete only when the sample size is enormously large such that the pseudo-random numbers from a full-period random number generator are exhausted. Kinematic formulation methods may result in solutions that are neither complete nor sound because of the numerical treatments. This is illustrated by an example of one-way clutch in Fig. 1. The known dimensional tolerances are the hub height $\mathbf{a} = [27.595, 27.695]$, the ring radius $\mathbf{e} = [50.7875, 50.8125]$, and the roller radius $\mathbf{r} = [11.42, 11.44]$. The variation of the roller position b needs to be estimated. By the direct linearization methods (DLM) with root-sum-square (RSS) and worst-case (WC) [1], we have $\mathbf{b}^{RSS} = [4.3585, 5.2625]$ and $\mathbf{b}^{WC} = [4.1368, 5.4842]$ respectively. However, the true variation range is $\mathbf{b} = [4.0838, 5.4405]$, which can be derived from the direct analysis of geometry. The combination of the largest a and r and the smallest e generates the lower bound of b. The combination of the smallest a and r and the largest e forms the upper bound of b. We can see $\mathbf{b}^{RSS}$ is sound but not complete, whereas $\mathbf{b}^{WC}$ is neither complete nor sound.

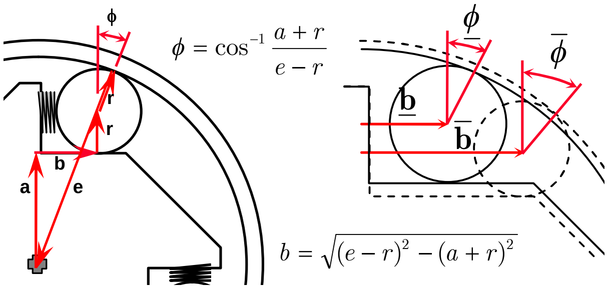


Fig. 1: An example of one-way clutch variation estimation.

Let $z = f(x_1, \ldots, x_n)$ be a general relation in tolerance analysis, where x_i's ($i = 1, \ldots, n$) are the variation source variables (inputs), and z is the performance variable (output). Let $[\underline{x}_i, \overline{x}_i]$'s ($i = 1, \ldots, n$) be the respective intervals of the input tolerances and $[\underline{z}, \overline{z}]$ a variation range estimate. $[\underline{z}, \overline{z}]$ is complete if and only if the following statement is true: "for any combination of inputs x_i's within the respective $[\underline{x}_i, \overline{x}_i]$'s, the output $z = f(x_1, \ldots, x_n)$ must be included in the estimated $[\underline{z}, \overline{z}]$". That is,

$$(\forall x_1 \in [\underline{x}_1, \overline{x}_1]) \cdots (\forall x_n \in [\underline{x}_n, \overline{x}_n])(\exists z \in [\underline{z}, \overline{z}])(f(x_1, \ldots, x_n) = z)$$

Similarly, the estimation is sound if and only if the following statement is true: "for any output z within the estimated $[\underline{z}, \overline{z}]$, there must exist a combination of inputs x_i's within the respective $[\underline{x}_i, \overline{x}_i]$'s such that $z = f(x_1, \ldots, x_n)$". That is,

$$(\forall z \in [\underline{z}, \overline{z}])(\exists x_1 \in [\underline{x}_1, \overline{x}_1]) \cdots (\exists x_n \in [\underline{x}_n, \overline{x}_n])(f(x_1, \ldots, x_n) = z)$$

For instance, in the one-way clutch example of Fig. 1, we are able to assert

$$(\forall a \in [27.595, 27.695])(\forall e \in [50.7875, 50.8125])(\forall r \in [11.42, 11.44])(\exists b \in [4.0838, 5.4405])\left(b = \sqrt{(e-r)^2 + (a+r)^2}\right)$$

and

$$(\forall b \in [4.0838, 5.4405])(\exists a \in [27.595, 27.695])(\exists e \in [50.7875, 50.8125])(\exists r \in [11.42, 11.44])\left(b = \sqrt{(e-r)^2 + (a+r)^2}\right)$$

Therefore, the logic interpretation of numerical results as above enables us to assess the completeness and soundness of range estimations. The attention of interpretability needs to be given in tolerance analysis. Recently, we proposed a new semantic tolerance modeling scheme [2, 3, 4] based on generalized intervals to enhance the interpretability of tolerance modeling. The purpose of semantic tolerance modeling is to embed logic relationships and engineering implications into the mathematical representation. With logical quantifiers, the relationship between tolerance specifications and implications of stacking may be derived from formulations. With the explicit differentiation between a priori and a posteriori tolerances, models can capture process-oriented semantics such as the difference between rigid and flexible materials in assemblies and the sequence of assembly.

In this paper, we present a semantic tolerance analysis approach based on interval vector loops to estimate tolerance accumulations. To ensure the interpretability of numerical results, new interpretable Jacobi algorithms and interpretable Hansen-Sengupta algorithms are developed to solve interval linear and nonlinear constraints. In the remainder of the paper, a brief review of vector loop based tolerance analysis methods and the generalized interval as the basis of semantic tolerance modeling are given in Section 2. In Section 3 the new interpretable linear system solver is constructed. In Section 4 the interpretable nonlinear system solver is described. The new analysis approach is illustrated with two numerical examples in Section 5.

2. BACKGROUND

There is a substantial amount of literature on tolerance modeling, analysis, and synthesis [5, 6, 7]. Here, we only give a brief overview of vector loop based analysis methods that are closely related to the proposed semantic tolerance analysis approach, as reviewed in Section 2.1. The main properties and notations of generalized intervals are summarized in Section 2.2. And the quantified interval constraint satisfaction problem is reviewed in Section 2.3.

2.1 Vector Loop-based Tolerance Analysis

Traditionally tolerance analysis is product-oriented. Dimensional limit, geometric variation, and kinematic displacement can be modeled mathematically in vectors and matrices. The vectorial tolerancing methods (Wirtz *et al*. [8], Martinsen [9]) model size, form, location, and orientation tolerances in a unified vector format in order to provide an integrated quality control loop. Rivest *et al*. [10] employed the kinematic characteristics of links between datum and toleranced features to model chains of variations. Clément *et al.* [11] identified and analyzed functional elements called TTRSs which are associated with geometric constraints. The small-displacement torsor methods (Bourdet and Ballot [12], Giordano and Duret [13], Descrochers [14]) approximate the rotation and translation displacement in the form of torsors. The matrix representation methods (Whitney *et al*. [15], Desrochers and Riviere [16], Lafond and Laperrière [17]) model small displacement in kinematic chains in the form of homogenous transformation matrices. Recently, Desrochers *et al.* [18] combined the torsor and matrix-based representations for tolerance analysis. Chase *et al*. [1, 19, 20] performed analysis of assemblies with tolerance vectors and small kinematic adjustments with linear approximations of implicit geometric constraints. Sacks and Joskowicz [21] analyzed 2D kinematic tolerances of assemblies with contact changes by the aid of contact constraints. Zou and Morse [22] proposed a fitting condition test method based on geometric constraints of gap closure between components.

In recent years, process-oriented analysis approaches were also proposed to consider the accumulation effects of manufacturing processes. With 1D vector loops, Zhang [23] combined the relation between functional requirements and dimensional tolerances with the one between dimensional and machining tolerances for simultaneous tolerancing. Based on constraints of force closure (Liu and Hu [24], Chang and Gossard [25]), 3D vector loops were used to predict variation accumulation in sheet metal joinings with the linearized finite element formulation. Long and Hu [26] extended the method to include the variation of fixtures during assembly operations. The single-station methods were also extended to multi-station approaches (Shiu *et al.* [27], Camelio *et al.* [28]) where variations are propagated in stages with tooling variations incorporated. Recently, Huang *et al.* [29] developed a stream-of-variation method to estimate dimensional variations in rigid-body assemblies for single-station and multi-station systems considering fixtures based on kinematic constraints.

In the above vector loop based methods, variation problems are formulated based on constraints of either form closure or force closure. The numerical treatments applied in these approaches prohibit interpretable numerical results. The main reason is that the commonly used solving methods with linearization and high-order approximations do not incorporate interpretability. During computation, the logic relationships among variables are left out. Therefore, the completeness and soundness of the results cannot be verified. In this paper, we propose a semantic tolerance analysis approach based on a new structure of interpretable system solvers. Generalized intervals are used for a unified variation representation.

2.2 Generalized Intervals

The semantic tolerance model is based on modal interval analysis (MIA) [30, 31, 32], which is an algebraic and semantic extension of the classic interval analysis (IA) [33]. A *modal interval* or *generalized interval* $\mathbf{x} := [\underline{x}, \overline{x}] \in \mathbb{KR}$ is called *proper* when $\underline{x} \le \overline{x}$ and *improper* when $\underline{x} \ge \overline{x}$. The set of proper intervals is denoted by $\mathbb{IR} = \{[\underline{x}, \overline{x}] \mid \underline{x} \le \overline{x}\}$ and the set of improper intervals by $\overline{\mathbb{IR}} = \{[\underline{x}, \overline{x}] \mid \underline{x} \ge \overline{x}\}$. The width of $\mathbf{x}$ is $\mathrm{wid}[\underline{x}, \overline{x}] := |\overline{x} - \underline{x}|$ and the center is found by $\mathrm{mid}[\underline{x}, \overline{x}] := (\underline{x} + \overline{x})/2$. A real function $f(x)$ where $x \in \mathbb{R}^n$ can be extended to $\mathbf{f}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{KR}^n$, which is called a $\mathbb{KR}$-extension, AE-extension, or modal extension. The real arithmetic is extended to the so-called Kaucher arithmetic [34].

Three special operators, *pro*, *imp*, and *dual*, are defined in the Kaucher arithmetic. Given a generalized interval $\mathbf{x} = [\underline{x}, \overline{x}] \in \mathbb{KR}$, $\mathrm{pro}\,\mathbf{x} := [\min(\underline{x}, \overline{x}), \max(\underline{x}, \overline{x})]$ and $\mathrm{imp}\,\mathbf{x} := [\max(\underline{x}, \overline{x}), \min(\underline{x}, \overline{x})]$ return the respective proper and improper interval values. $\mathrm{dual}[\underline{x}, \overline{x}] := [\overline{x}, \underline{x}]$ builds a relationship between proper and improper intervals. Related to the arithmetic operations $\circ \in \{+, -, \times, \div\}$, $(\mathrm{dual}\,\mathbf{x}) \circ (\mathrm{dual}\,\mathbf{y}) = \mathrm{dual}(\mathbf{x} \circ \mathbf{y})$. The *inclusion* relationship between modal intervals is defined as $[\underline{x}, \overline{x}] \subseteq [\underline{y}, \overline{y}] \Leftrightarrow (\underline{y} \le \underline{x}) \wedge (\overline{x} \le \overline{y})$. The *less than or equal to* relationship is defined as $[\underline{x}, \overline{x}] \le [\underline{y}, \overline{y}] \Leftrightarrow (\underline{x} \le \underline{y}) \wedge (\overline{x} \le \overline{y})$.

Different from IA, the group property is maintained in MIA because generalized intervals are closed under the Kaucher arithmetic operations. A generalized interval $\mathbf{a}$ is an *algebraic solution* of the equation $\mathbf{f}(\mathbf{x}) = \mathbf{b}$ where $\mathbf{x}$ is unknown if the original algebraic relation is still valid when the variable $\mathbf{x}$ is replaced by the interval result $\mathbf{a}$, i.e., $\mathbf{f}(\mathbf{a}) = \mathbf{b}$. This property is called *algebraic closure*. It simplifies the numerical analysis process while the interpretability is preserved.

Another uniqueness of generalized intervals is the modal semantic extension. Unlike IA which identifies an interval by a set of real numbers only, MIA identifies an interval by a set of predicates which is fulfilled by real numbers. Each interval $\mathbf{x} \in \mathbb{KR}$ has an associated logical quantifier, either existential ($\exists$) or universal ($\forall$). For a real relation $\phi(x) = z$ where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$, the semantics of its modal extension can be expressed with the quantifiers, which are derived based on the modalities of generalized intervals. As universal quantifiers precede existential ones, such quantified propositions have the form of

$$(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}})(\mathrm{Q}_z\, z \in \mathrm{pro}\,\mathbf{z})(\exists x_I \in \mathrm{pro}\,\mathbf{x}_I)(\phi(x) = z)$$

where $\mathcal{P}$ and I are disjoint sets of indices for proper and improper components of $\mathbf{x}_{\mathcal{P}\cup I} \in \mathbb{KR}^n$, $\mathrm{Q}_z = \forall$ if $\mathbf{z} \in \overline{\mathbb{IR}}$, and $\mathrm{Q}_z = \exists$ if $\mathbf{z} \in \mathbb{IR}$.

2.3 Quantified Interval Constraint Satisfaction

The constraint satisfaction problem over intervals had been studied in the past decade [35, 36]. Given a constraint $c(x_1,\dots,x_n)$, the main objective is to find a $(\mathbf{x}_1,\dots,\mathbf{x}_n) \in \mathbb{IR}^n$ to approximate the solution set $S \subset \mathbb{R}^n$ such that $(\exists x_1 \in \mathbf{x}_1)\cdots(\exists x_n \in \mathbf{x}_n)(c(x_1,\dots,x_n))$. Recently, universally quantified interval constraints received more attention, which is to find the solution $(\mathbf{x}_1,\dots,\mathbf{x}_n) \in \mathbb{IR}^n$ such that $(\forall x_1 \in \mathbf{x}_1)\cdots(\forall x_n \in \mathbf{x}_n)(c(x_1,\dots,x_n))$.

Quantifier elimination [37, 38] is the traditional method to solve quantified constraints. More recently, Benhamou et al. [39, 40] developed an inner contracting algorithm by splitting the complementary domain for box consistency. Shary [41, 32] formulated inner and outer estimation algorithms for generalized linear systems. Sainz et al. [42] developed linear and nonlinear algorithms to find algebraic solutions of constraints. Ratschan [43, 44] developed a branch-and-prune algorithm for inequality constraints over real numbers. Herrero et al. [45] developed a quantified set inversion algorithm for inner and outer estimations based on the interpretability of MIA evaluations. Goldsztejn and Jaulin [46] developed a branch-and-prune algorithm for inner and outer approximations of existentially quantified equality constraints.

The above quantified constraint solving methods concentrate on inner or outer approximations of solution sets. The interpretation requires that all variables are either universal or existential. In this paper, we generalize the quantified interval constraint satisfaction problem such that the numerical result is remained interpretable when the collection of variables is partially universal or existential.

3. SOLVING INTERPRETABLE LINEAR CONSTRAINTS

As mentioned in Section 2.1, the linearization approach used in the existing vector loop based tolerance analysis methods does not support interpretability. Thus the completeness and soundness of the numerical results cannot be verified. Here, we describe a new linearization and solving process that generates interpretable numerical results.

For $\mathbf{x} \in \mathbb{KR}^n$, a linear system of generalized intervals

$$\mathbf{A}\cdot\mathbf{x} = \mathbf{b} \tag{3.1}$$

where $\mathbf{A} = (\mathbf{a}_{ij})_{n\times n} \in \mathbb{KR}^{n\times n}$ and $\mathbf{b} \in \mathbb{KR}^n$, is closely associated with two inclusion relationships $\mathbf{A}\cdot\mathbf{x} \subseteq \mathbf{B}$ and $\mathbf{A}\cdot\mathbf{x} \supseteq \mathbf{B}$, given as

$$\mathbf{A}\cdot\mathbf{x} = \mathbf{B} \Leftrightarrow (\mathbf{A}\cdot\mathbf{x} \subseteq \mathbf{B}) \wedge (\mathbf{A}\cdot\mathbf{x} \supseteq \mathbf{B}) \tag{3.2}$$

If a *Jacobi interval operator* is defined as

$$\Im(\mathbf{x}_i) := \frac{\mathbf{b}_i - \sum_{i\neq j} \mathrm{dual}\,\mathbf{a}_{ij}\cdot \mathrm{dual}\,\mathbf{x}_j}{\mathrm{dual}\,\mathbf{a}_{ii}} \qquad (0 \notin \mathrm{pro}\,\mathbf{a}_{ii} \text{ and } i = 1,\cdots,n) \tag{3.3}$$

the following theorem provides the foundation to solve the linear system in Eqn.(3.1).

Theorem 3.1 [42] (*i*) If $\mathbf{x}$ is a solution to $\mathbf{A}\cdot\mathbf{x}\subseteq\mathbf{b}$, $\Im(\mathbf{x})$ is a solution to $\mathbf{A}\cdot\mathbf{x}\supseteq\mathbf{b}$. (*ii*) If $\mathbf{x}$ is a solution to $\mathbf{A}\cdot\mathbf{x}\supseteq\mathbf{b}$, $\Im(\mathbf{x})$ is a solution to $\mathbf{A}\cdot\mathbf{x}\subseteq\mathbf{b}$.

However, the linear system in Eqn.(3.1) is not interpretable if it includes multi-incident $\mathbf{x}_j$'s which are existential. Because the concatenation of $\exists x\in\mathbf{x}$ and $\exists x\in\mathbf{x}$ is not $\exists x\in\mathbf{x}$ in general. More formally

$$\left(\mathrm{Q}_{y_1}\, y_1\in\operatorname{pro}\mathbf{y}_1\right)(\exists x\in\operatorname{pro}\mathbf{x})\left(\mathrm{Q}_{z_1}\, z_1\in\operatorname{pro}\mathbf{z}_1\right)\left(z_1=f_1(x,y_1)\right)$$

and

$$\left(\mathrm{Q}_{y_2}\, y_2\in\operatorname{pro}\mathbf{y}_2\right)(\exists x\in\operatorname{pro}\mathbf{x})\left(\mathrm{Q}_{z_2}\, z_2\in\operatorname{pro}\mathbf{z}_2\right)\left(z_2=f_2(x,y_2)\right)$$

do not necessarily lead to

$$\left(\mathrm{Q}_{y_1}\, y_1\in\operatorname{pro}\mathbf{y}_1\right)\left(\mathrm{Q}_{y_2}\, y_2\in\operatorname{pro}\mathbf{y}_2\right)(\exists x\in\operatorname{pro}\mathbf{x})\left(\mathrm{Q}_{z_1}\, z_1\in\operatorname{pro}\mathbf{z}_1\right)\left(\mathrm{Q}_{z_2}\, z_2\in\operatorname{pro}\mathbf{z}_2\right)\left(z_1=f_1(x,y_1)\wedge z_2=f_2(x,y_2)\right)$$

To ensure interpretability, a transformed and interpretable linear system

$$\begin{cases}\mathbf{a}_{11}\mathbf{x}_1+\mathbf{a}_{12}\operatorname{pro}\mathbf{x}_2+\ldots+\mathbf{a}_{1n}\operatorname{pro}\mathbf{x}_n\subseteq\mathbf{b}_1\\ \mathbf{a}_{21}\operatorname{pro}\mathbf{x}_1+\mathbf{a}_{22}\mathbf{x}_2+\ldots+\mathbf{a}_{2n}\operatorname{pro}\mathbf{x}_n\subseteq\mathbf{b}_2\\ \cdots\\ \mathbf{a}_{n1}\operatorname{pro}\mathbf{x}_1+\mathbf{a}_{n2}\operatorname{pro}\mathbf{x}_2+\ldots+\mathbf{a}_{nn}\mathbf{x}_n\subseteq\mathbf{b}_n\end{cases}\tag{3.4}$$

or

$$\begin{cases}\mathbf{a}_{11}\mathbf{x}_1+\mathbf{a}_{12}\operatorname{imp}\mathbf{x}_2+\ldots+\mathbf{a}_{1n}\operatorname{imp}\mathbf{x}_n\supseteq\mathbf{b}_1\\ \mathbf{a}_{21}\operatorname{imp}\mathbf{x}_1+\mathbf{a}_{22}\mathbf{x}_2+\ldots+\mathbf{a}_{2n}\operatorname{imp}\mathbf{x}_n\supseteq\mathbf{b}_2\\ \cdots\\ \mathbf{a}_{n1}\operatorname{imp}\mathbf{x}_1+\mathbf{a}_{n2}\operatorname{imp}\mathbf{x}_2+\ldots+\mathbf{a}_{nn}\mathbf{x}_n\supseteq\mathbf{b}_n\end{cases}\tag{3.5}$$

should be solved instead, where each occurrence of the variables except the diagonal ones is transformed to its proper or improper counterpart in the new system. The notations of Eqn.(3.4) and Eqn.(3.5) are simplified as

$$\mathbf{A}\cdot\mathbf{x}^{pro}\subseteq\mathbf{b}\tag{3.6}$$

and

$$\mathbf{A}\cdot\mathbf{x}^{imp}\supseteq\mathbf{b}\tag{3.7}$$

The algebraic solutions can be interpreted as

$$\left(\forall a_P\in\mathbf{a}_P\right)\left(\forall b_I\in\operatorname{pro}\mathbf{b}_I\right)\left(\forall x_P\in\mathbf{x}_P\right)\left(\exists a_I\in\operatorname{pro}\mathbf{a}_I\right)\left(\exists b_P\in\mathbf{b}_P\right)\left(\exists x_I\in\operatorname{pro}\mathbf{x}_I\right)\left(A\cdot x=b\right)\tag{3.8}$$

and

$$\left(\forall a_I\in\operatorname{pro}\mathbf{a}_I\right)\left(\forall b_P\in\mathbf{b}_P\right)\left(\forall x_I\in\operatorname{pro}\mathbf{x}_I\right)\left(\exists a_P\in\mathbf{a}_P\right)\left(\exists b_I\in\operatorname{pro}\mathbf{b}_I\right)\left(\exists x_P\in\mathbf{x}_P\right)\left(A\cdot x=b\right)\tag{3.9}$$

respectively.

An enhanced interpretable Jacobi algorithm is developed to solve Eqn.(3.6), as listed in Fig. 2, where the Jacobi operator is applied to the original and the transformed variables alternately. We define a *proper transform Jacobi interval operator* as

$$\Im^{pro}(\mathbf{x}_i):=\frac{\mathbf{b}_i-\sum_{i\neq j}\operatorname{dual}\mathbf{a}_{ij}\cdot\operatorname{imp}\mathbf{x}_j}{\operatorname{dual}\mathbf{a}_{ii}}\qquad\left(0\notin\operatorname{pro}\mathbf{a}_{ii}\text{ and } i=1,\cdots,n\right)\tag{3.10}$$

Applying the Jacobi operator in Eqn.(3.3) to the transformed variable $\mathbf{x}^{pro}$ is equivalent to applying the proper transform Jacobi operator in Eqn.(3.10) to the original variable $\mathbf{x}$.

Similarly, an interpretable Jacobi algorithm to solve Eqn.(3.7) is listed in Fig. 3, where an *improper transform Jacobi interval operator* is defined as

$$\mathfrak{I}^{imp}(\mathbf{x}_i) := \frac{\mathbf{b}_i - \sum_{i \neq j} \operatorname{dual} \mathbf{a}_{ij} \cdot \operatorname{pro} \mathbf{x}_j}{\operatorname{dual} \mathbf{a}_{ii}} \quad \left(0 \notin \operatorname{pro} \mathbf{a}_{ii} \text{ and } i = 1, \cdots, n \right) \tag{3.11}$$

Function $\mathbf{x} = interpretableJacobi\_pro(\mathbf{A}, \mathbf{b})$

1. Initial estimation of $\mathbf{y}^{(0)}$ as the point (real) solution of $\operatorname{mid}(\mathbf{A})x = \operatorname{mid}(\mathbf{b})$ such that $\operatorname{imp} \mathbf{A} \cdot \mathbf{y}^{(0)} \subseteq \operatorname{pro} \mathbf{b}$;
2. $\mathbf{x}^{(0)} = \mathfrak{I}(\mathbf{y}^{(0)})$ associated with $\operatorname{imp} \mathbf{A} \cdot \mathbf{x} = \operatorname{pro} \mathbf{b}$, which is also the initial solution to $\mathbf{A} \cdot \mathbf{x} \supseteq \mathbf{b}$ as $k = 0$;
3. Iterate the follows associated with $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ until $\mathbf{x}$ converges:
   $k = k + 1$;
   If k is odd, $\mathbf{x}^{(k)} = \mathfrak{I}^{pro}(\mathbf{x}^{(k-1)})$;
   otherwise, $\mathbf{x}^{(k)} = \mathfrak{I}(\mathbf{x}^{(k-1)})$.

Fig. 2: Proper transform Jacobi algorithm for the linear system in Eqn.(3.4).

Function $\mathbf{x} = interpretableJacobi\_imp(\mathbf{A}, \mathbf{b})$

1. Initial estimation of $\mathbf{y}^{(0)}$ as the point (real) solution of $\operatorname{mid}(\mathbf{A})x = \operatorname{mid}(\mathbf{b})$ such that $\operatorname{pro} \mathbf{A} \cdot \mathbf{y}^{(0)} \supseteq \operatorname{imp} \mathbf{b}$;
2. $\mathbf{x}^{(0)} = \mathfrak{I}(\mathbf{y}^{(0)})$ associated with $\operatorname{pro} \mathbf{A} \cdot \mathbf{x} = \operatorname{imp} \mathbf{b}$, which is also the initial solution to $\mathbf{A} \cdot \mathbf{x} \subseteq \mathbf{b}$ as $k = 0$;
3. Iterate the follows associated with $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ until $\mathbf{x}$ converges:
   $k = k + 1$;
   If k is odd, $\mathbf{x}^{(k)} = \mathfrak{I}^{imp}(\mathbf{x}^{(k-1)})$;
   otherwise, $\mathbf{x}^{(k)} = \mathfrak{I}(\mathbf{x}^{(k-1)})$.

Fig. 3: Improper transform Jacobi algorithm for the linear system in Eqn.(3.5).

Theorem 3.2 (*i*) If $\mathbf{x}$ is a solution to $\mathbf{A} \cdot \mathbf{x} \supseteq \mathbf{b}$, then $\mathbf{x}$ is also a solution to $\mathbf{A} \cdot \mathbf{x}^{pro} \supseteq \mathbf{b}$. (*ii*) If $\mathbf{x}$ is a solution to $\mathbf{A} \cdot \mathbf{x} \subseteq \mathbf{b}$, then $\mathbf{x}$ is also a solution to $\mathbf{A} \cdot \mathbf{x}^{imp} \subseteq \mathbf{b}$.

Proof. (*i*) Because of the monotonicity of inclusion in the Kaucher arithmetic, i.e., $\mathbf{x} \supseteq \mathbf{x}' \wedge \mathbf{y} \supseteq \mathbf{y}' \Rightarrow (\mathbf{x} \circ \mathbf{y}) \supseteq (\mathbf{x}' \circ \mathbf{y}')$ where $\circ \in \{+, -, \times, \div\}$. For any $\mathbf{u}, \mathbf{v} \in \mathbb{KR}$, $\operatorname{pro} \mathbf{v} \supseteq \mathbf{v}$ and $\mathbf{u} \operatorname{pro} \mathbf{v} \supseteq \mathbf{uv}$. Thus $\operatorname{pro} \mathbf{x}_j \supseteq \mathbf{x}_j$ for $j = 1, \ldots, n$, and $\sum_{j=1}^{n} \mathbf{a}_{ij} \operatorname{pro} \mathbf{x}_j \supseteq \sum_{j=1}^{n} \mathbf{a}_{ij} \mathbf{x}_j$ for $i = 1, \ldots, n$. Therefore, $\mathbf{A} \cdot \mathbf{x}^{P} \supseteq \mathbf{A} \cdot \mathbf{x} \supseteq \mathbf{b}$. (*ii*) can similarly be proved. ∎

In Fig. 2, at the ($2k$)th step in the iterative solving process, applying the Jacobi operator in Eqn.(3.3) to $\mathbf{x}^{(2k-1)}$, we receive $\mathbf{x}^{(2k)} = \mathfrak{I}(\mathbf{x}^{(2k-1)})$ such that $\mathbf{A} \cdot \mathbf{x}^{pro(2k)} \supseteq \mathbf{A} \cdot \mathbf{x}^{(2k)} \supseteq \mathbf{b}$. Then at the ($2k+1$)th step, applying the proper transform Jacobi operator in Eqn.(3.10) to $\mathbf{x}^{(2k)}$, we have $\mathbf{x}^{(2k+1)} = \mathfrak{I}(\mathbf{x}^{pro(2k)})$ such that $\mathbf{A} \cdot \mathbf{x}^{(2k+1)} \subseteq \mathbf{b}$. The iteration continues until stopping criteria are met. If $\mathbf{x}^{(k)}$ converges to $\mathbf{x}^{\infty}$, $\mathbf{x}^{\infty}$ is a solution to the interpretable linear system in Eqn.(3.4). The iteration of the algorithm in Fig. 3 is similar.

The interpretable linear system solving algorithms in Fig. 2 and Fig. 3 ensure the interpretability of numerical results. This is regarded as an important step towards interpretable tolerance analysis. Based on this, an interpretable nonlinear constraint solver is developed in Section 4.

4. SOLVING INTERPRETABLE NONLINEAR CONSTRAINTS

In general, we would like to find the solution to a nonlinear system $\mathbf{f} : \mathbb{KR}^m \times \mathbb{KR}^n \to \mathbb{KR}^n$ of generalized intervals

$$\mathbf{f}(\mathbf{u}, \mathbf{x}) = 0 \tag{4.1}$$

where $\mathbf{u} \in \mathbb{KR}^m$ and $\mathbf{x} \in \mathbb{KR}^n$. The correspondingly desirable interpretation of the solution $\mathbf{x}$ is

$$(\forall u_{\mathcal{P}} \in \mathbf{u}_{\mathcal{P}})(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}})(\exists u_I \in \operatorname{pro}\mathbf{u}_I)(\exists x_I \in \operatorname{pro}\mathbf{x}_I)(f(u,x) = 0) \tag{4.2}$$

The *Hansen-Sengupta* (HS) *operator* is one of the most used methods to estimate the outer enclosure of nonlinear systems, which is based on the *interval Gauss-Seidel* (GS) *operator*

$$\Gamma(\mathbf{A}, \mathbf{b}, \mathbf{x}_i) := \frac{\mathbf{b}_i - \sum_{i \neq j} \mathbf{a}_{ij} \cdot \mathbf{x}_j}{\mathbf{a}_{ii}} \cap \mathbf{x}_i \quad (\, 0 \notin \mathbf{a}_{ii} \text{ and } i = 1, \cdots, n) \tag{4.3}$$

for the classic interval linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A} \in \mathbb{IR}^{n \times n}$, $\mathbf{b} \in \mathbb{IR}^n$, and $\mathbf{x} \in \mathbb{IR}^n$. If a *Lipschitz matrix* $\mathbf{J} \in \mathbb{IR}^{n \times n}$ with respect to $\mathbf{x}$ for a continuous function $f(x) = 0$ can be found such that

$$f(x) - f(y) \in \mathbf{J}(x - y) \quad (\text{for all } x, y \in \mathbf{x}) \tag{4.4}$$

then the HS operator is defined as

$$\mathrm{H}(f, \mathbf{J}, C, \mathbf{x}, x_0) := x_0 + \Gamma(C\mathbf{J}, -Cf(x_0), \mathbf{x} - x_0) \tag{4.5}$$

where $x_0 \in \mathbf{x}$ and $C \in \mathbb{R}^{n \times n}$ is the preconditioning matrix that usually is $(\operatorname{mid}\mathbf{J})^{-1}$. In particular, an interval evaluation of the continuous function's Jacobi matrix is an instance of Lipschitz matrices.

The guaranteed existence of the solution from the HS operator has been proved and summarized in the following theorem.

Theorem 4.1 ([47] Theorem 5.1.8) If the HS operator in Eqn.(4.5) is applied in contraction, then (*i*) $\mathbf{x} \cap f^{-1}(0) \subseteq \mathrm{H}(f, \mathbf{J}, C, \mathbf{x}, x_0)$; (*ii*) If $\mathbf{x} \cap \mathrm{H}(f, \mathbf{J}, C, \mathbf{x}, x_0) = \varnothing$, then f contains no zeros in $\mathbf{x}$; (*iii*) If $x_0 \in \operatorname{int}(\mathbf{x})$ and $\varnothing \neq \mathrm{H}(f, \mathbf{J}, C, \mathbf{x}, x_0) \subseteq \operatorname{int}(\mathbf{x})$, then $\mathbf{J}$ is strongly regular and f contains a unique zero in $\mathrm{H}(f, \mathbf{J}, C, \mathbf{x}, x_0)$.

Now we show the extension of the classic HS operator for our interpretable nonlinear systems in Eqn.(4.1). If a Lipschitz matrix $\mathbf{J} = [\mathbf{J}^u(\operatorname{pro}\mathbf{x}), \mathbf{J}^x(\operatorname{pro}\mathbf{u})]$ where $\mathbf{J}^u : \mathbb{IR}^n \to \mathbb{IR}^{n \times m}$ and $\mathbf{J}^x : \mathbb{IR}^m \to \mathbb{IR}^{n \times n}$ can be found such that

$$f(u,x) - f(u_0, x_0) \in \mathbf{J}^u(u - u_0) + \mathbf{J}^x(x - x_0) \quad (\text{for } \forall u \in \operatorname{pro}\mathbf{u} \text{ and } \forall x \in \operatorname{pro}\mathbf{x}) \tag{4.6}$$

then

$$f(u,x) \in f(u_0, x_0) + \mathbf{J}^u(\operatorname{pro}\mathbf{u} - u_0) + \mathbf{J}^x(\operatorname{pro}\mathbf{x} - x_0) \quad (\text{for } \forall u \in \operatorname{pro}\mathbf{u} \text{ and } \forall x \in \operatorname{pro}\mathbf{x}) \tag{4.7}$$

If we replace the interval GS operator in Eqn.(4.5) by the proper transform Jacobi interval operator in Eqn.(3.10), a *proper transform HS operator* is then defined as

$$\mathrm{H}_{\forall\exists}^{pro}(f, \mathbf{J}^x, C, \mathbf{x}, x_0) := x_0 + \mathfrak{I}^{pro}(C \operatorname{pro}\mathbf{J}^x, -Cf(u_0, x_0) - C \operatorname{dual}(\operatorname{pro}\mathbf{J}^u \cdot (\mathbf{u} - u_0)), \mathbf{x} - x_0) \tag{4.8}$$

where $C = (\operatorname{mid}\mathbf{J}^x)^{-1}$. Similarly, an *improper transform HS operator* is defined as

$$\mathrm{H}_{\forall\exists}^{imp}(f, \mathbf{J}^x, C, \mathbf{x}, x_0) := x_0 + \mathfrak{I}^{imp}(C \operatorname{imp}\mathbf{J}^x, -Cf(u_0, x_0) - C \operatorname{dual}(\operatorname{imp}\mathbf{J}^u \cdot (\mathbf{u} - u_0)), \mathbf{x} - x_0) \tag{4.9}$$

Theorem 4.2 (*i*) The solution of the nonlinear system Eqn.(4.1) by the proper transform HS operator in Eqn.(4.8) is interpreted as

$$(\forall u_{\mathcal{P}} \in \mathbf{u}_{\mathcal{P}})(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}})(\exists g^u \in \mathbf{J}^u)(\exists g^x \in \mathbf{J}^x)(\exists u_I \in \operatorname{pro}\mathbf{u}_I)(\exists x_I \in \operatorname{pro}\mathbf{x}_I)(f(u,x) = 0) \tag{4.10}$$

(*ii*) The solution by the improper transform HS operator in Eqn.(4.9) is interpreted as

$$(\forall u_I \in \operatorname{pro}\mathbf{u}_I)(\forall x_I \in \operatorname{pro}\mathbf{x}_I)(\exists g^u \in \mathbf{J}^u)(\exists g^x \in \mathbf{J}^x)(\exists u_{\mathcal{P}} \in \mathbf{u}_{\mathcal{P}})(\exists x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}})(f(u,x) = 0) \tag{4.11}$$

Proof.

(*i*) If $\mathbf{x}$ is an algebraic solution of the linearized system $C \operatorname{pro}\mathbf{J}^x \cdot (\mathbf{x}^{pro} - x_0) \subseteq -Cf(u_0, x_0) - C \operatorname{dual}(\operatorname{pro}\mathbf{J}^u \cdot (\mathbf{u} - u_0))$, which is equivalent to $f(u_0, x_0) + \operatorname{pro}\mathbf{J}^u \cdot (\mathbf{u} - u_0) + \operatorname{pro}\mathbf{J}^x \cdot (\mathbf{x}^{pro} - x_0) \subseteq 0$, then the interpretation of the solution is

$$\left(\forall u_{\mathcal{P}} \in \mathbf{u}_{\mathcal{P}}\right)\left(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}}\right)\left(\forall g^{u} \in \mathbf{J}^{u}\right)\left(\forall g^{x} \in \mathbf{J}^{x}\right)\left(\exists u_{I} \in \operatorname{pro}\mathbf{u}_{I}\right)\left(\exists x_{I} \in \operatorname{pro}\mathbf{x}_{I}\right) \\ \left(f(u_0,x_0) + g^{u}(u-u_0) + g^{x}(x-x_0) = 0\right) \tag{4.12}$$

Since the interpretation of Eqn.(4.7) is

$$(\forall u \in \operatorname{pro}\mathbf{u})(\forall x \in \operatorname{pro}\mathbf{x})\left(\exists g^{u} \in \mathbf{J}^{u}\right)\left(\exists g^{x} \in \mathbf{J}^{x}\right)\left(f(u,x) = f(u_0,x_0) + g^{u}(u-u_0) + g^{x}(x-x_0)\right) \tag{4.13}$$

the concatenation of Eqn.(4.12) and Eqn.(4.13) yields the interpretation in Eqn.(4.10).

(*ii*) If $\mathbf{X}$ is an algebraic solution of the linearized system $C\operatorname{imp}\mathbf{J}^{x}\cdot(\mathbf{x}^{imp}-x_0) \supseteq -Cf(u_0,x_0) - C\operatorname{dual}\left(\operatorname{imp}\mathbf{J}^{u}\cdot(\mathbf{u}-u_0)\right)$, which is equivalent to $f(u_0,x_0) + \operatorname{imp}\mathbf{J}^{u}\cdot(\mathbf{u}-u_0) + \operatorname{imp}\mathbf{J}^{x}\cdot(\mathbf{x}^{imp}-x_0) \supseteq 0$, then the interpretation of the solution is

$$\left(\forall u_{I} \in \operatorname{pro}\mathbf{u}_{I}\right)\left(\forall x_{I} \in \operatorname{pro}\mathbf{x}_{I}\right)\left(\forall g^{u} \in \mathbf{J}^{u}\right)\left(\forall g^{x} \in \mathbf{J}^{x}\right)\left(\exists u_{\mathcal{P}} \in \mathbf{u}_{\mathcal{P}}\right)\left(\exists x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}}\right) \\ \left(f(u_0,x_0) + g^{u}(u-u_0) + g^{x}(x-x_0) = 0\right) \tag{4.14}$$

The concatenation of Eqn.(4.14) and Eqn.(4.13) yields the interpretation in Eqn.(4.11). □

The interpretable nonlinear system solving algorithms based on the proper and improper transform HS operators are listed in Fig. 4 and Fig. 5 respectively. With the interpretable solvers, the completeness and soundness of numerical estimations in tolerance analysis can be verified. This is illustrated by examples in Section 5.

Function $\mathbf{x} = interpretableHS\_pro(f,\mathbf{u})$

1. Initial guess of $\mathbf{x}^{(0)}$ such that $\forall u \in \operatorname{pro}\mathbf{u}, \exists x \in \operatorname{pro}\mathbf{x}^{(0)}, f(u,x) = 0$;
2. Calculate $\mathbf{J}_u = \operatorname{pro}\left.\frac{\partial f}{\partial u}\right|_{\mathbf{x}}$, $\mathbf{J}_x = \operatorname{pro}\left.\frac{\partial f}{\partial x}\right|_{\mathbf{u}}$, $C = \left(\operatorname{mid}\mathbf{J}_x\right)^{-1}$, $u_0 = \operatorname{mid}\mathbf{u}$, $x_0 = \operatorname{mid}\mathbf{x}$, $\mathbf{A} = C\cdot\operatorname{pro}\mathbf{J}_x$, and $\mathbf{b} = C\cdot[-f(u_0,x_0) - \operatorname{dual}(\operatorname{pro}\mathbf{J}_u\cdot(\mathbf{u}-u_0))]$;
3. $\mathbf{x} = interpretableJacobi\_pro(\mathbf{A},\mathbf{b}) + x_0$;
4. Go to Step 2 until $\mathbf{x}$ converges.

Fig. 4: Proper transform HS algorithm for the nonlinear system in Eqn.(4.1).

Function $\mathbf{x} = interpretableHS\_imp(f,\mathbf{u})$

1. Initial guess of $\mathbf{x}^{(0)}$ such that $\forall u \in \operatorname{pro}\mathbf{u}, \exists x \in \operatorname{pro}\mathbf{x}^{(0)}, f(u,x) = 0$;
2. Calculate $\mathbf{J}_u = \operatorname{pro}\left.\frac{\partial f}{\partial u}\right|_{\mathbf{x}}$, $\mathbf{J}_x = \operatorname{pro}\left.\frac{\partial f}{\partial x}\right|_{\mathbf{u}}$, $C = \left(\operatorname{mid}\mathbf{J}_x\right)^{-1}$, $u_0 = \operatorname{mid}\mathbf{u}$, $x_0 = \operatorname{mid}\mathbf{x}$, $\mathbf{A} = C\cdot\operatorname{imp}\mathbf{J}_x$, and $\mathbf{b} = C\cdot[-f(u_0,x_0) - \operatorname{dual}(\operatorname{imp}\mathbf{J}_u\cdot(\mathbf{u}-u_0))]$;
3. $\mathbf{x} = interpretableJacobi\_imp(\mathbf{A},\mathbf{b}) + x_0$;
4. Go to Step 2 until $\mathbf{X}$ converges.

Fig. 5: Improper transform HS algorithm for the nonlinear system in Eqn.(4.1).

5. NUMERICAL EXAMPLES

5.1 Stacked Block Assembly

Fig. 6 shows an example of a stacked block assembly including a base, a rectangular plate and a cylindrical rod. With the known size tolerances of manufactured components, the kinematic variations of the assembly can be calculated with three interval vector loops. Each of the closed loops defines the algebraic relations between the size and kinematic variations. The vector components in each 2D translational or rotational direction sum up to zero, as listed in Tab. 1.

The tolerance analysis problem is formulated by nine constraints. It can be solved by using the improper transform HS algorithm in Fig. 5. The size tolerances (a,b,c,d,e,f) are assigned to be proper, and the resulted kinematic variations (u_1,u_2,u_3,u_4,u_5) and angular variations ($\phi_1,\phi_2,\phi_3,\phi_4$) are improper. Based on the interpretability rule in Eqn.(4.11), the result is interpreted as: for all the possible values within the kinematic variation ranges, there exists a combination of size tolerances such that the nonlinear constraints are satisfied. In other words, the kinematic variation estimations are sound. Generating verifiable numerical results is the main advantage of solving interpretable systems compared to the traditional analysis methods, where estimations are not interpretable. Consequently completeness or soundness of the estimations is unknown.

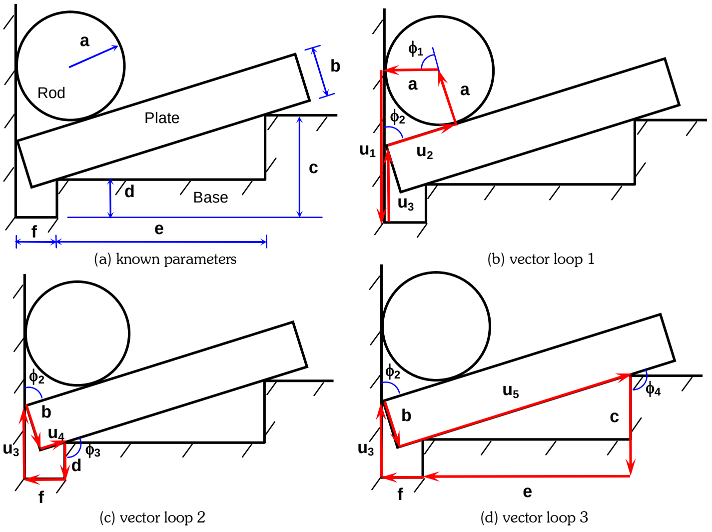


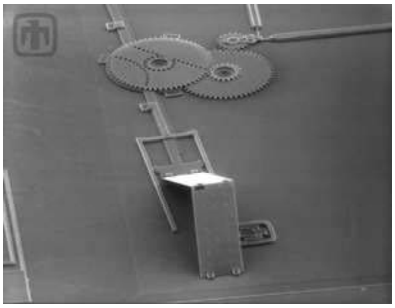
Fig. 6: A stacked block assembly with closed loops of variations.

Known Size Variations	$a = 6.62 \pm 0.2$ $\quad b = 6.805 \pm 0.075$ $\quad c = 10.675 \pm 0.125$ $d = 4.06 \pm 0.15$ $\quad e = 24.22 \pm 0.35$ $\quad f = 3.905 \pm 0.125$
Unknown Kinematic Variations	$u_1 = 18.7181 \pm ?$ $\quad u_2 = 8.6705 \pm ?$ $\quad u_3 = 10.0477 \pm ?$ $\quad u_4 = 2.1894 \pm ?$ $\quad u_5 = 27.2965 \pm ?$ $\phi_1 = 74.7243 \pm ?$ $\quad \phi_2 = -74.7243 \pm ?$ $\quad \phi_3 = -105.2761 \pm ?$ $\quad \phi_4 = -105.2761 \pm ?$
Loop 1	$\begin{cases} F_1 = u_2 \cos(90+\phi_2) + a\cos(180+\phi_2) + a\cos(180+\phi_1+\phi_2) = 0 \\ F_2 = u_3 + u_2 \sin(90+\phi_2) + a\sin(180+\phi_2) + a\sin(180+\phi_1+\phi_2) - u_1 = 0 \\ F_3 = 90+\phi_2+90+\phi_1+90+90-360 = 0 \end{cases}$
Loop 2	$\begin{cases} F_4 = b\cos(\phi_2) + u_4\cos(\phi_2+90) + d\cos(\phi_2+90+\phi_3) - f = 0 \\ F_5 = u_3 + b\sin(\phi_2) + u_4 \sin(\phi_2+90) + d\sin(\phi_2+90+\phi_3) = 0 \\ F_6 = 90+\phi_2-90+90+\phi_3-90+180 = 0 \end{cases}$
Loop 3	$\begin{cases} F_7 = b\cos(\phi_2) + u_5\cos(\phi_2+90) + c\cos(\phi_2+90+\phi_4) - e - f = 0 \\ F_8 = u_3 + b\sin(\phi_2) + u_5 \sin(\phi_2+90) + c\sin(\phi_2+90+\phi_4) = 0 \\ F_9 = 90+\phi_2-90+90+\phi_4-90+180 = 0 \end{cases}$
Result of the Improper Transform HS Algorithm	$\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \mathbf{u}_5 \end{bmatrix} = \begin{bmatrix} [19.2237, 18.2128] \\ [9.1142, 8.2268] \\ [10.3430, 9.7524] \\ [2.4497, 1.9291] \\ [27.8137, 26.7794] \end{bmatrix}$ $\begin{bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \\ \boldsymbol{\varphi}_3 \\ \boldsymbol{\varphi}_4 \end{bmatrix} = \begin{bmatrix} [75.9385, 73.5092] \\ [-73.5092, -75.9385] \\ [-104.0615, -106.4908] \\ [-104.0615, -106.4908] \end{bmatrix}$
Interpretation of the Result	$(\forall \phi_1 \in [73.5092, 75.9385])(\forall \phi_2 \in [-75.9385, -73.5092])(\forall \phi_3 \in [-106.4908, -104.0615])(\forall \phi_4 \in [-106.4908, -104.0615])$ $(\forall u_1 \in [18.2128, 19.2237])(\forall u_2 \in [8.2268, 9.1142])(\forall u_3 \in [9.7524, 10.3430])(\forall u_4 \in [1.9291, 2.4497])(\forall u_5 \in [26.7794, 27.8137])$ $(\exists a \in [6.42, 6.82])(\exists b \in [6.73, 6.88])(\exists c \in [10.55, 10.8])(\exists d \in [3.91, 4.21])(\exists e \in [23.87, 24.57])(\exists f \in [3.78, 4.03])$ The above constraints are satisfied.

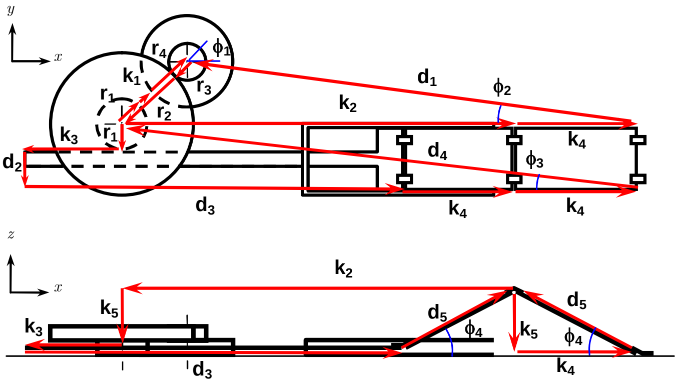
Tab. 1: The numerical formulation of the stacked block assembly in Fig. 6.

5.2 MEMS Mirror Assembly

Fig. 7 shows a second example, which is a polysilicon micro mirror mechanism. The assembly includes transmission gears, a linear rack, pop-up mirrors, and hinges. Constraints are formulated based on five closed loops, as shown in Fig. 7(b). Tab. 2 lists the size and kinematic variables, constraints, and the numerical result. The system was solved by using the proper transform HS algorithm in Fig. 4. The interpretation is based on the interpretability rule in Eqn.(4.10).



(a) Polysilicon Micro Mirror Assembly (Courtesy of Sandia National Laboratories, SUMMiT[TM] Technologies, mems.sandia.gov)



(b) Five closed loops of interval vectors

Fig. 7: The MEMS mirror assembly with closed loops of variations.

Known Variations	$r_1 = 0.655 \pm 0.003$ $r_2 = 1.945 \pm 0.003$ $r_3 = 0.475 \pm 0.003$ $r_4 = 1.225 \pm 0.003$ $d_1 = 12.50 \pm 0.05$ $d_2 = 1.145 \pm 0.003$ $d_3 = 10.40 \mp 0.05$ $d_4 = 14.30 \pm 0.05$ $d_5 = 3.45 \pm 0.05$ $\phi_4 = 32 \pm 0.5$
Unknown Variations	$k_1 = 0.54 \pm ?$ $k_2 = 11.2 \pm ?$ $k_3 = 2.0 \pm ?$ $k_4 = 2.9 \pm ?$ $k_5 = 1.8 \pm ?$ $\phi_1 = 42 \pm ?$ $\phi_2 = 7 \pm ?$ $\phi_3 = 7 \pm ?$
Constraints	$r_1 + k_1 + r_4 - r_3 - r_2 = 0$ $k_2 + k_4 - d_1 \cos\phi_2 - (r_2 + r_3)\cos\phi_1 = 0$ $d_1 \sin\phi_2 - (r_2 + r_3)\sin\phi_1 = 0$ $-k_3 + d_3 + 2k_4 - d_4 \cos\phi_3 = 0$ $-r_1 - d_2 + d_4 \sin\phi_3 = 0$ $-k_3 + d_3 + d_5 \cos\phi_4 - k_2 = 0$ $-k_5 + d_5 \sin\phi_4 = 0$ $k_4 - d_5 \cos\phi_4 = 0$
Result of the Proper Transform HS Algorithm	$\mathbf{k}_1 = [0.5520, 0.5280]$ $\mathbf{k}_2 = [11.5874, 10.9336]$ $\mathbf{k}_3 = [2.2836, 1.8470]$ $\mathbf{k}_4 = [2.9843, 2.8672]$ $\mathbf{k}_5 = [1.8811, 1.7754]$ $\boldsymbol{\varphi}_1 = [56.2146, 28.2061]$ $\boldsymbol{\varphi}_2 = [8.1120, 6.8354]$ $\boldsymbol{\varphi}_3 = [7.2812, 7.1813]$
Interpretation of the Result	$(\forall r_1 \in [0.652, 0.658])(\forall r_2 \in [1.942, 1.948])(\forall r_3 \in [0.472, 0.478])(\forall r_4 \in [1.222, 1.228])$ $(\forall d_1 \in [12.45, 12.55])(\forall d_2 \in [1.142, 1.148])(\forall d_4 \in [14.25, 14.35])(\forall d_5 \in [3.40, 3.50])(\forall \phi_4 \in [31.5, 32.5])$ $(\exists k_1 \in [0.5280, 0.5520])(\exists k_2 \in [10.9336, 11.5874])(\exists k_3 \in [1.8470, 2.2836])(\exists k_4 \in [2.8672, 2.9843])(\exists k_5 \in [1.7754, 1.8811])$ $(\exists \phi_1 \in [28.2061, 56.2146])(\exists \phi_2 \in [6.8354, 8.1120])(\exists \phi_3 \in [7.1813, 7.2812])(\exists d_3 \in [10.35, 10.45])$ The above constraints are satisfied.

Tab. 2: The numerical formulation of the MEMS mirror assembly in Fig. 7.

In general, different modality assignments of tolerance intervals lead to different numerical results and different interpretations. Depending on the designer's intention or desired semantics, the corresponding combinations of proper and improper intervals can be applied.

6. CONCLUDING REMARKS

Semantic tolerance modeling is to enrich tolerance modeling and analysis structures such that more process-oriented tolerancing semantics and intents can be embedded in mathematical representations. The ultimate goal is to support better design and manufacturing specifications. In this paper, we presented a semantic tolerance analysis approach by solving quantified interval constraints. To ensure the interpretability of numerical results, new interpretable Jacobi algorithms are developed to solve linear constraints. The interpretable relations among variables can be maintained during computation. New proper and improper transform HS algorithms are also developed to solve nonlinear constraints. Based on logic relationships, completeness and soundness of numerical results can be verified. Generating verifiable numerical results is the main advantage of solving interpretable systems compared to the traditional analysis methods where results are not interpretable.

The future work includes the extension of the current approach for under- and over-constrained systems. With the logical quantifiers, it is possible to convert under- or over- constrained systems to well-constrained ones. The convergence conditions of nonlinear system solvers and how the preconditioning affects the interpretation need to be investigated further. The developed interpretable methods can also be extended to other applications. Since the developed numerical methods are generic in nature, they could potentially be applied in other engineering domains such as robust control and prediction under uncertainties.

7. REFERENCES

[1] Chase, K. W.; Gao, J.; Magleby, S. P.: General 2-D tolerance analysis of mechanical assemblies with small kinematic adjustments, J. of Design and Manufacturing, 5(4), 1995, 263-274.

[2] Wang, Y.: Semantic tolerance modeling based on modal interval analysis, In Proc. 2006 NSF Workshop on Reliable Engineering Computing (REC'06), Georgia Institute of Technology, Savannah, GA, 2006, 293-318.

[3] Wang, Y.: Semantic tolerance modeling, In Proc. 2006 ASME International Design Engineering Technical Conferences & The Computer and Information in Engineering Conference (IDETC2006), Philadelphia, PA, 2006, Paper No. DETC2006-99069.

[4] Wang, Y.: Semantic tolerancing with generalized intervals, Computer-Aided Design & Applications, 4(1-4), 2007, 257-266.

[5] Hong, Y. S.; Chang, T.-C.: A comprehensive review of tolerancing research, Int. J. of Production Research, 40(11), 2002, 2425-2459.

[6] Zhang, H. C. (eds.): Advanced Tolerancing Techniques, John Wiley & Sons, New York, 1997.

[7] Shah, J. J.; Ameta, G.; Shen, Z.; Davidson, J.: Navigating the tolerance analysis maze, Computer-Aided Design & Applications, 4(5), 2007, 705-718.

[8] Wirtz, A.; Gachter, C.; Wipf, D.: From unambiguously defined geometry to the perfect quality control loop, Annals of the CIRP, 42(1), 1993, 615-618.

[9] Martinsen, K.: Statistical process control using vectorial tolerancing, In CIRP/JSPE/ASME Proc. 4th CIRP Seminar on Computer Aided Tolerancing, University of Tokyo, Tokyo, Japan, 1995, 195-210.

[10] Rivest, L.; Fortin, C.; Morel, C.: Tolerancing a solid model with a kinematic formulation, Computer-Aided Design, 26(6), 1994, 465-476.

[11] Clément, A.; Valade, C.; Rivière, A.: The TTRSs: 13 oriented constraints for dimensioning, tolerancing and inspection, Advanced Mathematical Tools in Metrology III, 1996, 24-41.

[12] Bourdet, P.; Ballot, E.: Geometric behavior for computer aided tolerancing, In CIRP/JSPE/ASME Proc. 4th CIRP Seminar on Computer Aided Tolerancing, University of Tokyo, Tokyo, Japan, 1995, 143-154.

[13] Giordano, M.; Duret, D.: Clearance space and deviation space, In CIRP Proc. 3rd Seminars on Computer Aided Tolerancing, Cachan, France, 1993, 179-196.

[14] Desrochers, A.: Modeling three-dimensional tolerance zones using screw parameters, Proc. 25th ASME Design Automation Conference, 1999, Paper No. DAC-8587.

[15] Whitney, D. E.; Gilbert, O. L.; Jastrzebski, M.: Representation of geometric variations using matrix transforms for statistical tolerance analysis in assemblies, Research in Engineering Design, 6, 1994, 191-210.

[16] Desrochers, A.; Riviere, A.: A matrix approach to the representation of tolerance zones and clearances, Int. J. Advanced Manufacturing Technology, 13, 1997, 630-636.

[17] Lafond, P.; Laperrière, L.: Jacobian-based modeling of dispersions affecting pre-defined functional requirements of mechanical assemblies, Proc. IEEE Int. Symp. Assembly & Task Planning, 1999, 20-25.

[18] Descrochers, A.; Ghie, W.; Laperrière, L.: Application of a unified Jacobian-torsor model for tolerance analysis, ASME J. Comp. in Sci. Eng., 3(1), 2003, 2-14.

[19] Chase, K. W.; Magleby, S. P.; Gao, J.; Sorensen, C. D.: Including geometric feature variations in tolerance analysis of mechanical assemblies, IIE Transactions, 28(10), 1996, 795-807.

[20] Gao, J.; Chase, K. W.; Magleby, S. P.: Generalized 3-D tolerance analysis of mechanical assemblies with small kinematic adjustments, IIE Transactions, 30(4), 1998, 367-377.

[21] Sacks, E.; Joskowicz, L.: Parametric kinematic tolerance analysis of general planar systems, Computer-Aided Design, 30(9), 1998, 707-714.

[22] Zou, Z.; Morse, E.P.: A gap-based approach to capture fitting conditions for mechanical assembly, Computer-Aided Design, 36(8), 2004, 691-700.

[23] Zhang, G.: Simultaneous tolerancing for design and manufacturing, Intl J. Prod. Res., 34(12), 1996, 3361-3382.

[24] Liu, S.; Hu, S.: Variation simulation for compliant sheet metal assemblies using finite element methods, ASME J. of Manufacturing Science & Engineering, 119(3), 1997, 368-374.

[25] Chang, M.; Gossard, D. C.: Modeling the assembly of compliant, non-ideal parts, Computer-Aided Design, 29(10), 1997, 701-708.

[26] Long, Y.; Hu, S.: A unified model for variation simulation of sheet metal assemblies, In Geometric Design Tolerancing: Theories, Standards, and Applications, ed. by H.A. Elmaraghy, London: Chapman & Hall, 1998, pp.208-219

[27] Shiu, B. W.; Ceglarek, D.; Shi, J.: Muilti-stations sheet metal assembly modeling and diagnostics, Trans. NAMRI, 24, 1996, 199-204.

[28] Camelio, J.; Hu, S. J.; Ceglarek, D.: Modeling variation propagation of multi-station assembly systems with compliant parts, ASME J. of Mechanical Design, 125(4), 2003, 673-681.

[29] Huang, W.; Lin, J.; Kong, Z.; Ceglarek, D.: Stream-of-variation (SOVA) modeling – Part II: A generic 3D variation model for rigid body assembly in multistation assembly processes, ASME J. Manu. Sci. & Eng., 129(4), 2007, 832-842.

[30] Gardenes, E.; Sainz, M. A.; Jorba, L.; Calm, R.; Estela, R.; Mielgo, H.; Trepat, A.: Modal intervals, Reliable Computing, 7(2), 2001, 77-111.

[31] Dimitrova, N. S.; Markov, S. M.; Popova, E. D.: Extended interval arithmetics: new results and applications, in L. Atanassova; J. Herzberger (Eds.): Computer Arithmetic and Enclosure Methods, Elsevier, 1992, 225-232.

[32] Shary, S. P.: A new technique in systems analysis under interval uncertainty and ambiguity, Reliable Computing, 8(2), 2002, 321-418.

[33] Moore, R. E.: Interval Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1966.

[34] Kaucher, E.: Interval analysis in the extended interval space IR, Computing Supplementum, 2, 1980, 33-49.

[35] Hyvönen, E.: Constraint reasoning based on interval arithmetic: the tolerance propagation approach, Artificial Intelligence, 58, 1992, 71-112.

[36] Benhamo, F.; McAllester, D.; Van Hentenryck, P.: CLP(Intervals) revisited, Proc. ILPS'94, 1994, pp.124-138.

[37] Collins, G. E.: Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition, Lecture Notes in Computer Science, 33, 1975, 134-183.

[38] Collins, G. E.; Hong, H.: Partial cylindrical algebraic decomposition for quantifier elimination, J. of Symbolic Computing, 12, 1991, 299-328.

[39] Benhamou, F.; Goualard, F.: Universally quantified interval constraints, Proc. CP2000, Lecture Notes in Computer Science, 1894, 2000, 67-82.

[40] Benhamou, F.; Goualard, F.; Languenou, E.; Christie, M.: Interval constraint solving for camera control and motion planning, ACM Trans. Computational Logic, 5(4), 2004, 732-767.

[41] Shary, S. P.: Interval Gauss-Seidel method for generalized solution sets to interval linear systems, Reliable Computing, 7, 2001, 141-155.

[42] Sainz, M. A.; Gardenes, E.; Jorba, L.: Formal solution to systems of interval linear or non-linear equations, Reliable Computing, 8(3), 2002, 189-211.

[43] Ratschan, S.: Continuous first-order constraint satisfaction, In Artificial Intelligence, Automated Reasoning, and Symbolic Computation, ed. By J. Calmet, B. Benhamou, O. Caprotti, L. Henocque, and V. Sorge, Lecture Notes in Computer Science, 2385, 2002, 181-195.
[44] Ratschan, S.: Efficient solving of quantified inequality constraints over the real numbers, ACM Trans. on Computational Logic, 7(4), 2006, 723-748.
[45] Herrero, P.; Sainz, M. A.; Vehi, J.; Jaulin, L.: Quantified set inversion with applications to control, Reliable Computing, 11(5), 2005, 369-382.
[46] Goldztejn, A.; Jaulin, L.: Inner and outer approximations of existentially quantified equality constraints, Proc. CP2006, Lecture Notes in Computer Science, 4204, 2006, 198-212.
[47] Neumaier, A.: Interval Methods for Systems of Equations, University Press, Cambridge, 1990.