Ribs and Fans of Bézier Curves and Surfaces

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ABSTRACT

Ribs and fans are interesting geometric entities that can be derived from a Bézier curve or surface. A rib is a Bézier curve whose control points are linear combination of the given curve. A fan is a vector field defined between two ribs. The degree of ribs and fans are lower than the given curve. The ribs and fans of a Bézier surface can be similarly derived as linear combinations. We present methods to transform the control points of a given Bezier curve or surface into the control points and vectors of its ribs and fans. Then, we show that a Bézier curve of degree *n* is decomposed into a rib of degree (*n*-1) and a fan of degree (*n*-2). We also show that a Bézier surface of degree (*m*, *n*) is decomposed into a rib of degree (*m*-1, *n*-1) and three fans of degrees (*m*-1, *n*-2), (*m*-2, *n*-1), and (*m*-2, *n*-2), respectively. We also illustrate some of design examples.

Keywords: rib, fan, Bézier curve and surface, decomposition

1. INTRODUCTION

Bézier curves and surfaces are one of the most basic and widely accepted theories of CAGD. We find numerous implementations of this theory in almost any GUI-based software application and versatile devices: from a professional CAD modeling tool running on a grid-computing environment to a trendy Flash[5] viewer running on mobile devices. (Bézier is ubiquitous!) This kind of popularity is far surpassing its powerful siblings such as NURBS due to its relatively rich expressiveness and structural elegance based on numerous nice properties found through extensive previous research works. To name a few, we can list famous properties such as affine invariance, convex hull property, variation diminishing property [1-2],[4]. In this paper, we present new properties of Bézier curves and surfaces: ribs and fans. Then, we show that a Bézier curve or surface can be decomposed into ribs and fans based on the theorems proposed in the paper.

A rib itself is a Bézier curve or surface with a lower degree than the given curve or surface. A fan is a vector field whose degree is lower than its origin, and defined between two ribs of different degrees. We present methods to transform the control points of the given curve or surface into rib control points and fan control vectors. Then, we show that a Bézier curve of degree n can be decomposed into a rib of degree (*n*-1) and a fan of degree (*n*-2). We also show that a Bézier surface of degree (*m*, *n*) is decomposed into a rib of degree (*m*-1, *n*-1) and three fans of degrees (*m*-1, *n*-2), (*m*-2, *n*-1), and (*m*-2, *n*-2), respectively. The lengths of the fans are further controlled by scalar functions. We present relevant notations and definitions, introduce theories, and illustrate some of design examples.

We organize the rest of this paper as follows. In Section 2, we first review the definitions of Bézier curve and surfaces, and some properties of Bernstein polynomials. In Section 3, we describe the ribs and fans of Bézier curves and surfaces in detail. In Section 4, we illustrate some of design examples showing the geometric features of ribs and fans. In Section 5, we conclude this paper with short remarks.

2. PRELIMINARIES

A Bézier curve $\mathbf{b}(t)$ of degree n is defined as a parametric linear combination of control points $\mathbf{b}_i$ at t: $\mathbf{b}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(t)$ where weighting factors $B_i^n(t)$ are i-th Bernstein polynomials of degree n. A Bézier surface $\mathbf{b}(u,v)$ of degree (m,n) is defined as a tensor product interpolant as follows [1]: $\mathbf{b}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}\mathbf{b}_{i,j}B_i^m(u)B_j^n(v)$ where u and v are two independent parameters defined over real numbers. Each Bernstein polynomial is recursively expressed as follows:

$$B_i^{n+1}(t)=(1-t)B_i^n(t)+tB_{i-1}^n(t), \tag{1}$$

Eqn. (1) states the recursion relation between two set of Bernstein polynomials of degrees different by one. If we want a recursion formula for bigger differences, we can repeatedly apply Eqn. (1) to the terms of lower degrees. For example, we can get the relation between $B_*^{n+1}(t)$ and $B_*^{n-1}(t)$, whose degrees are different by two, by applying one more recursion step as follows:

$$B_i^{n+1}(t)=(1-t)^2B_i^{n-1}(t)+2(1-t)tB_{i-1}^{n-1}(t)+t^2B_{i-2}^{n-1}(t)=\sum_{j=0}^{2}B_j^2(t)B_{i-j}^{n-1}(t). \tag{2}$$

We introduce two additional recursion relations between two set of Bernstein polynomials of degrees different by one as follows [1],[3]:

$$B_i^n(t)=\frac{n}{n-i}(1-t)B_i^{n-1}(t),\quad 0\le i<n. \tag{3}$$

$$B_i^n(t)=\frac{n}{i}tB_{i-1}^{n-1}(t),\quad 0<i\le n. \tag{4}$$

3. RIBS AND FANS

3.1. Curve Case

We introduce the concept of ribs and fans of a Bézier curve, and describe the definitions of rib control points and fan control vectors. Then, we present a method to decompose a given Bézier curve into a rib and a fan.

3.1.1. Rib and Its Control Points

Let $\mathbf{b}_i\equiv\mathbf{r}_i^n$ $(0\le i\le n)$ be the control points of the given Bézier curve $\mathbf{b}(t)$ of degree *n*. We recursively define *rib control points* of degree k as follows:

$$\mathbf{r}_i^k=\frac{(k-i)}{k}\mathbf{r}_i^{k+1}+\frac{i}{k}\mathbf{r}_{i+1}^{k+1}\quad\text{for }1\le k\le n-1 \tag{5}$$

These rib control points define a *rib* as a Bézier curve of degree *k*:

$$\mathbf{r}^k(t)=\sum_{i=0}^{k}\mathbf{r}_i^kB_i^k(t)\quad\text{for }1\le k\le n, \tag{6}$$

Hence, a rib $\mathbf{r}^k(t)$ is a Bézier curve that is defined by a *rib ancestor* (or a higher rib) $\mathbf{r}^{k+1}(t)$ for $(1\le k<n)$ and the transformation rule in Eqn. (5). See Fig. 1. The rib of the lowest degree is a line segment $\mathbf{r}^1(t)$ passing through two rib control points $\mathbf{r}_0^1=\mathbf{b}_0$ and $\mathbf{r}_1^1=\mathbf{b}_n$. For convenience, we define that the given Bézier curve itself is the rib of the highest degree: $\mathbf{r}^n(t)\equiv\mathbf{b}(t)$. Note that any rib of degree *k* ($1\le k<n$) always connects the two ends of the $\mathbf{b}(t)$.

3.1.2. Fan and Its Control Vectors

We define a *fan control vector* $\mathbf{f}_i^k$ of degree *k* using three consecutive rib control points of degree (*k*+2):

$$\mathbf{f}_i^k=\mathbf{r}_{i+1}^{k+2}-\frac{1}{2}\left(\mathbf{r}_i^{k+2}+\mathbf{r}_{i+2}^{k+2}\right)\quad\text{for }0\le k\le n-2. \tag{7}$$

Now, we define a *fan* using fan control vectors $\mathbf{f}_i^k$ $(0\le i\le k)$ as follows:

$$\mathbf{f}^k(t)=\sum_{i=0}^{k}\mathbf{f}_i^kB_i^k(t)\quad\text{for }(0\le k\le n-2). \tag{8}$$

A fan has the form of a Bézier curve. However, it is not the combination of points but the vectors that have special geometric meanings. See Fig. 2. Note that, basically, we can derive a fan $\mathbf{f}^k(t)$ from a rib $\mathbf{r}^{k+2}(t)$. The fan $\mathbf{f}^0(t)$ of the lowest degree is a fixed vector defined by three control points.

3.1.3. Theory

Before presenting the general theory, we explain how ribs and fans are defined for a cubic Bézier curve $\mathbf{b}(t) \equiv \mathbf{r}^3(t)$ in Fig. 1(a). Using Eqn. (5), we can define control points for three ribs in Fig. 1(b): $\{\mathbf{r}_i^3 \equiv \mathbf{b}_i\}$ in blue, $\{\mathbf{r}_i^2\}$ in purple, and $\{\mathbf{r}_i^1\}$ in red. Specially, $\mathbf{r}_1^2 = \left(\mathbf{r}_1^3 + \mathbf{r}_2^3\right)/2$ in purple. Using Eqn. (6) and rib control points, we can draw three rib curves in Fig. 1(c): $\mathbf{r}^3(t)$ in blue, $\mathbf{r}^2(t)$ in purple and $\mathbf{r}^1(t)$ in red. Fig. 1(d) illustrates sampled vectors from a sequence of scaled fans $2t(1-t)\mathbf{f}^1(t)$ and $2t(1-t)\mathbf{f}^0(t)$. (It is equal to overlap of Fig. 2(d) and 2(h). Details are explained below) Circles are points placed on a rib and represents end points of scaled fan vectors. In Fig. 1(e), the internal curves are *fan curves* whose control polygon corresponds to a sequence of sampled fan vectors at the same value of parameter *t*: i.e., $\mathbf{r}^3(t_j)$, $\mathbf{r}^2(t_j)$, and $\mathbf{r}^1(t_j)$.

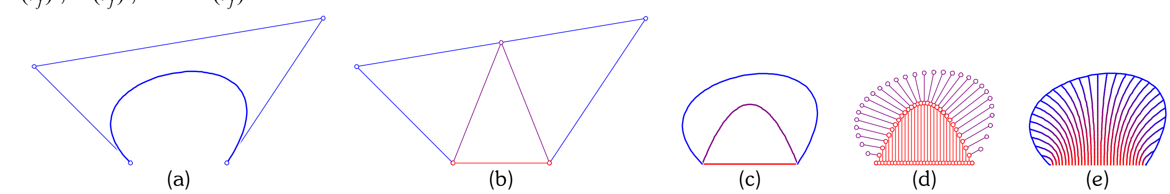


Fig. 1. Generation of ribs and fans for a cubic Bézier curve. Read above for details.

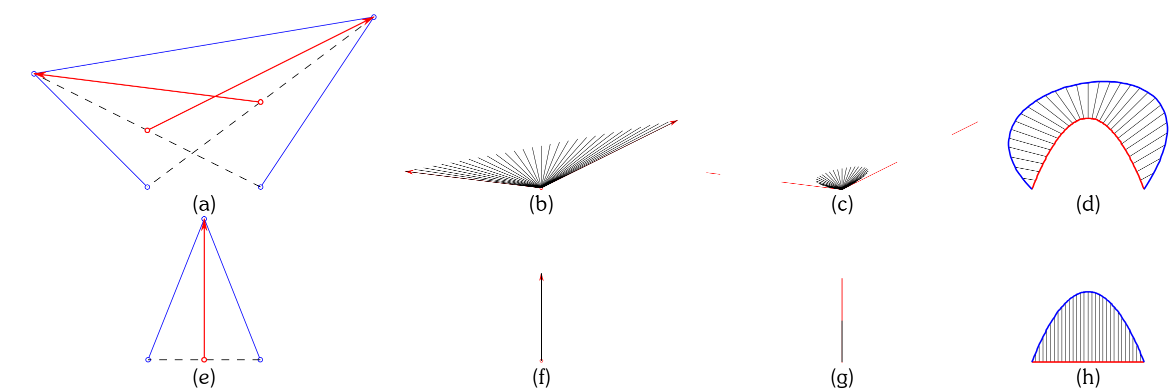


Fig. 2. Detailed procedures for generation of fans for a cubic Bézier curve in Fig.1. Read below for details.

Fig. 2 summarizes procedures to generate fan vector field $\mathbf{f}^k(t)$. First, in Fig. 2(a)-(d), we show procedures to get a fan from the rib $\mathbf{r}^3(t)$. Using Eqn. (7), we derive two fan control vectors (in red) $\mathbf{f}_0^1$ and $\mathbf{f}_1^1$ from the rib control points $\mathbf{r}_i^3$ as shown in Fig. 2(a). (For example, $\mathbf{f}_0^1 = \mathbf{r}_1^3 - \left(\mathbf{r}_0^3 + \mathbf{r}_2^3\right)/2$.) Using Eqn. (8), we generate a fan $\mathbf{f}^1(t)$ by interpolating the fan control vectors. (Here, $\mathbf{f}^1(t) = (1-t)\mathbf{f}_0^1 + t\mathbf{f}_1^1$.) The sampled vectors (in black) of $\mathbf{f}^1(t)$ is shown in Fig. 2(b). If we scale $\mathbf{f}^1(t)$ by $2t(1-t)$, we get $2t(1-t)\mathbf{f}^1(t)$ as in Fig. 2(c). According to the Theorem 1 (below), the scaled fan $2t(1-t)\mathbf{f}^1(t)$ is equal to the difference between ribs $\mathbf{r}^3(t)$ (in blue) and $\mathbf{r}^2(t)$ (in red.): i.e., $\mathbf{r}^3(t) - \mathbf{r}^2(t) = 2t(1-t)\mathbf{f}^1(t)$. Hence, as in Fig. 2(d), if we add the scaled fan to the lower rib $\mathbf{r}^2(t)$, the sum coincide with the upper rib $\mathbf{r}^3(t)$. In Fig. 2(e)-(h), we show similar procedures to get a simpler fan of the rib $\mathbf{r}^2(t)$. Fig. 2(e) shows the single fan control vector $\mathbf{f}_0^0$ using $\mathbf{r}_i^2$. As in Fig. 2(f), the fan is constant: $\mathbf{f}^0(t) = \mathbf{f}_0^0$. If we scales it as $2t(1-t)\mathbf{f}^0(t)$, the sampled vectors are collinear as in Fig. 2(g). As in Fig. 2(h), $\mathbf{r}^2(t) = \mathbf{r}^1(t) + 2t(1-t)\mathbf{f}^0(t)$. (We recommend more examples in the web site[6].)

Now, we present a general method to decompose an arbitrary Bézier curve into a rib and a fan. Specially, we show that the sum of lower rib and the scaled fan is equal to the upper rib or the given Bézier curve.

Theorem 1. (Rib and Fan of Bézier Curve) A Bézier curve $\mathbf{b}(t) \equiv \mathbf{r}^n(t)$ of degree $n \geq 2$ is decomposed into a rib $\mathbf{r}^{n-1}(t)$ of degree (n-1) and a fan $\mathbf{f}^{n-2}(t)$ of degree (n-2) as follows:

$$\mathbf{b}(t) = \mathbf{r}^n(t) = \mathbf{r}^{n-1}(t) + 2t(1-t)\mathbf{f}^{n-2}(t). \tag{9}$$

Proof> We prove the theorem by showing the equality in Eqn. (9) holds generally, based on mathematical induction. As a base step, when $n=2$, the given control points of $\mathbf{r}^2(t)$ are given as $\{\mathbf{b}_i = \mathbf{r}_i^2\}$ for $(0 \leq i \leq 2)$. By Eqn. (5), the control points $\{\mathbf{r}_i^1\}$ of the rib $\mathbf{r}^1(t)$ are expressed as follows: $\mathbf{r}_0^1 = \mathbf{r}_0^2$ and $\mathbf{r}_1^1 = \mathbf{r}_2^2$. Thus the base rib $\mathbf{r}^1(t)$ can be expressed as follows:

$$\mathbf{r}^1(t) = \sum_{i=0}^{1} \mathbf{r}_i^1 B_i^1(t) = (1-t)\mathbf{r}_0^2 + t\mathbf{r}_2^2. \tag{10}$$

By Eqn. (7), the fan control vector of degree 0 is defined as follow: $\mathbf{f}_0^0 = \mathbf{r}_1^2 - \frac{1}{2}\left(\mathbf{r}_0^2 + \mathbf{r}_2^2\right)$. Hence, the fan of degree 0 is identical to the control vector itself:

$$\mathbf{f}^0(t) = \mathbf{f}_0^0 = \mathbf{r}_1^2 - \frac{1}{2}\left(\mathbf{r}_0^2 + \mathbf{r}_2^2\right). \tag{11}$$

By applying Eqns. (10) and (11) to Eqn. (9), we can show that the theorem holds for the base step:

$$\mathbf{r}^1(t) + 2t(1-t)\mathbf{f}^0(t) = (1-t)\mathbf{r}_0^2 + t\mathbf{r}_2^2 + 2t(1-t)\left(\mathbf{r}_1^2 - \frac{1}{2}\left(\mathbf{r}_0^2 + \mathbf{r}_2^2\right)\right) = (1-t)^2\mathbf{r}_0^2 + 2t(1-t)\mathbf{r}_1^2 + t^2\mathbf{r}_2^2 = \sum_{i=0}^{2} \mathbf{r}_i^2 B_i^2(t) = \mathbf{r}^2(t) \tag{12}$$

Now, we assume that the induction hypothesis holds for $n=k$ as follows:

$$\mathbf{r}^k(t) = \mathbf{r}^{k-1}(t) + 2t(1-t)\mathbf{f}^{k-2}(t). \tag{13}$$

In the below, we show that the induction hypothesis of Eqn. (13) also holds for $n=k+1$ as follows:

$$\mathbf{r}^{k+1}(t) = \mathbf{r}^k(t) + 2t(1-t)\mathbf{f}^{k-1}(t). \tag{14}$$

Note that we can represent a Bézier curve of degree $(k+1)$ using the Bernstein polynomials of degree $(k-1)$ based on the property expressed in Eqn. (2). Hence, a rib $\mathbf{r}^{k+1}(t)$ of degree (k+1) can be expressed as follows:

$$\mathbf{r}^{k+1}(t) = \sum_{i=0}^{k+1} \mathbf{r}_i^{k+1} B_i^{k+1}(t) = \sum_{i=0}^{k+1} \mathbf{r}_i^{k+1}\left(\sum_{j=0}^{2} B_j^2(t) B_{i-j}^{k-1}(t)\right). \tag{15}$$

Using the definition of rib control points of Eqn. (5) and the properties of Bernstein polynomials in Eqns. (3) and (4), a rib $\mathbf{r}^k(t)$ of degree k can be expressed using the control points of $\mathbf{r}^{k+1}(t)$ of degree (k+1) and Bernstein polynomial of degree (k-1):

$$\begin{aligned}
\mathbf{r}^k(t) &= \sum_{i=0}^{k} \mathbf{r}_i^k B_i^k(t) = \frac{1}{k}\sum_{i=0}^{k}\left((k-i)\mathbf{r}_i^{k+1} + i\mathbf{r}_{i+1}^{k+1}\right)B_i^k(t) = \frac{1}{k}\left(\sum_{i=0}^{k}(k-i)\mathbf{r}_i^{k+1}B_i^k(t)\right) + \frac{1}{k}\left(\sum_{i=0}^{k} i\mathbf{r}_{i+1}^{k+1}B_i^k(t)\right)\\
&= \frac{1}{k}\left(\sum_{i=0}^{k-1}(k-i)\mathbf{r}_i^{k+1}\frac{k}{k-i}(1-t)B_i^{k-1}(t)\right) + \frac{1}{k}\left(\sum_{i=1}^{k} i\mathbf{r}_{i+1}^{k+1}\frac{k}{i}tB_{i-1}^{k-1}(t)\right) = \sum_{i=0}^{k-1}\mathbf{r}_i^{k+1}(1-t)B_i^{k-1}(t) + \sum_{i=0}^{k-1}\mathbf{r}_{i+2}^{k+1}tB_i^{k-1}(t)\\
&= \sum_{i=0}^{k-1}\left((1-t)\mathbf{r}_i^{k+1} + t\mathbf{r}_{i+2}^{k+1}\right)B_i^{k-1}(t)
\end{aligned} \tag{16}$$

Using the definitions in Eqns. (7) and (8), the expression of a fan $\mathbf{f}^{k-1}(t)$ in Eqn. (14) can be expanded as follows:

$$\mathbf{f}^{k-1}(t) = \sum_{i=0}^{k-1} \mathbf{f}_i^{k-1} B_i^{k-1}(t) = \sum_{i=0}^{k-1}\left(\mathbf{r}_{i+1}^{k+1} - \frac{1}{2}\left(\mathbf{r}_i^{k+1} + \mathbf{r}_{i+2}^{k+1}\right)\right)B_i^{k-1}(t). \tag{17}$$

Eqns. (16) and (17) state that both a rib curve of degree k and a fan of degree $(k-1)$ can be expressed using the Bernstein polynomials of the same degree $(k-1)$ and the same set of rib control points of degree $(k+1)$. We can apply Eqns. (16) and (17) to Eqn. (14) and evaluate it as follows:

$$\mathbf{r}^k(t)+2t(1-t)\mathbf{f}^{k-1}(t)=\sum_{i=0}^{k-1}\left((1-t)\mathbf{r}_i^{k+1}+t\mathbf{r}_{i+2}^{k+1}\right)B_i^{k-1}(t)+2t(1-t)\sum_{i=0}^{k-1}\left(\mathbf{r}_{i+1}^{k+1}-\frac{1}{2}\left(\mathbf{r}_i^{k+1}+\mathbf{r}_{i+2}^{k+1}\right)\right)B_i^{k-1}(t)$$
$$=\sum_{i=0}^{k-1}\left((1-t)^2\mathbf{r}_i^{k+1}+2t(1-t)\mathbf{r}_{i+1}^{k+1}+t^2\mathbf{r}_{i+2}^{k+1}t\right)B_i^{k-1}(t)=\sum_{i=0}^{k-1}\left(\sum_{j=0}^{2}\mathbf{r}_{i+j}^{k+1}B_j^2(t)\right)B_i^{k-1}(t). \tag{18}$$

We can further expand the last term of Eqn. (18) as follows:

$$\mathbf{r}^k(t)+2t(1-t)\mathbf{f}^{k-1}(t)=\sum_{i=0}^{k-1}\left(\sum_{j=0}^{2}\mathbf{r}_{i+j}^{k+1}B_j^2(t)\right)B_i^{k-1}(t)$$
$$=\begin{bmatrix}
\mathbf{r}_0^{k+1}B_0^2(t)B_0^{k-1}(t) & + & \mathbf{r}_1^{k+1}B_1^2(t)B_0^{k-1}(t) & + & \mathbf{r}_2^{k+1}B_2^2(t)B_0^{k-1}(t) & & & & & & \\
 & + & \mathbf{r}_1^{k+1}B_0^2(t)B_1^{k-1}(t) & + & \mathbf{r}_2^{k+1}B_1^2(t)B_1^{k-1}(t) & + & \cdots & & & & \\
 & & & + & \mathbf{r}_2^{k+1}B_0^2(t)B_2^{k-1}(t) & + & \cdots & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & \cdots & + & \mathbf{r}_{k-1}^{k+1}B_2^2(t)B_{k-3}^{k-1}(t) & & & \\
 & & & & & \cdots & + & \mathbf{r}_{k-1}^{k+1}B_1^2(t)B_{k-2}^{k-1}(t) & + & \mathbf{r}_k^{k+1}B_2^2(t)B_{k-2}^{k-1}(t) & \\
 & & & & & & + & \mathbf{r}_{k-1}^{k+1}B_0^2(t)B_{k-1}^{k-1}(t) & + & \mathbf{r}_k^{k+1}B_1^2(t)B_{k-1}^{k-1}(t) & + \ \mathbf{r}_{k+1}^{k+1}B_2^2(t)B_{k-1}^{k-1}(t)
\end{bmatrix} \tag{19}$$

Observation of the 3rd and 4th columns of the right hand side of Eqn. (19) leads to the simplification as follows:

$$\mathbf{r}^k(t)+2t(1-t)\mathbf{f}^{k-1}(t)=\mathbf{r}_0^{k+1}\left(\sum_{j=0}^{2}B_j^2(t)B_{0-j}^{k-1}(t)\right)+\mathbf{r}_1^{k+1}\left(\sum_{j=0}^{2}B_j^2(t)B_{1-j}^{k-1}(t)\right)+\cdots+\mathbf{r}_i^{k+1}\left(\sum_{j=0}^{2}B_j^2(t)B_{i-j}^{k-1}(t)\right)+\cdots$$
$$+\mathbf{r}_k^{k+1}\left(\sum_{j=0}^{2}B_j^2(t)B_{k-j}^{k-1}(t)\right)+\mathbf{r}_{k+1}^{k+1}\left(\sum_{j=0}^{2}B_j^2(t)B_{k+1-j}^{k-1}(t)\right)$$
$$=\sum_{i=0}^{k+1}\mathbf{r}_i^{k+1}\left(\sum_{j=0}^{2}B_j^2(t)B_{i-j}^{k-1}(t)\right) \tag{20}$$

This derivation is based on the fact that Bernstein polynomials of indices outside the proper range vanishes: i.e., $B_i^k(t)=0$ if $i\notin[0,k]$. Finally, if we apply Eqn. (2) into Eqn. (20), we get the equality of the induction step in Eqn. (15):

$$\mathbf{r}^k(t)+2t(1-t)\mathbf{f}^{k-1}(t)=\sum_{i=0}^{k+1}\mathbf{r}_i^{k+1}\left(\sum_{j=0}^{2}B_j^2(t)B_{i-j}^{k-1}(t)\right)=\sum_{i=0}^{k+1}\mathbf{r}_i^{k+1}B_i^{k+1}(t)=\mathbf{r}^{k+1}(t). \tag{21}$$

Hence, we can conclude that the induction hypothesis of Eqn. (13) holds for $n=k+1$. **Q.E.D.**

Corollary 1. A Bézier curve $\mathbf{b}(t)\equiv\mathbf{r}^n(t)$ of degree $n\ge 2$ can be decomposed into a single rib of degree l and a sequence of $(n-l)$ fans of degrees from $(n-2)$ to $(l-1)$ as follows:

$$\mathbf{b}(t)=\mathbf{r}^n(t)=\mathbf{r}^l(t)+2t(1-t)\sum_{k=l-1}^{n-2}\mathbf{f}^k(t),\quad 1\le l\le n-1. \tag{22}$$

We can derive Corollary 1 by repeatedly applying Theorem 1 to the subsequent ribs. (We skip the rigorous proof.) For example, $\mathbf{r}^n(t)$ can be decomposed in various ways according the degree of the rib chosen:

$$\mathbf{r}^n(t)=\mathbf{r}^{n-1}(t)+2t(1-t)\mathbf{f}^{n-2}(t)=\mathbf{r}^{n-2}(t)+2t(1-t)\left(\mathbf{f}^{n-2}(t)+\mathbf{f}^{n-3}(t)\right)=\mathbf{r}^{n-3}(t)+2t(1-t)\left(\mathbf{f}^{n-2}(t)+\mathbf{f}^{n-3}(t)+\mathbf{f}^{n-4}(t)\right) \tag{23}$$

Specially, the given curve can be decomposed into a line segment $\mathbf{r}^1(t)$ and a sequence of $(n-1)$ fan vectors. We have shown this example in Fig. 2 for the cubic case.

3.2. Surface Case

Now, we introduce the concept of ribs and fans of a Bézier surface. First, we define some notations and transformation rules of control points to derive rib control points and fan control vectors. Then, we present a method to decompose a given Bézier surface into a rib and three fans.

3.2.1. Notations and Transformation Rules

First, control points of a Bézier surface are placed in a $(m+1)\times(n+1)$ mesh (or a control net), and denoted as $\mathbf{r}_{i,j}^{m,n} \equiv \mathbf{b}_{i,j}$, where $0 \le i \le m$ and $0 \le j \le n$. Then, a Bézier surface of degree $(m,\ n)$ is denoted as follows:

$$\mathbf{r}^{m,n}(u,v) \equiv \mathbf{b}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{r}_{i,j}^{m,n} \cdot B_i^m(u) \cdot B_j^n(v)\,. \tag{24}$$

With these new notations, we express the decomposition of Bézier curves that belong to the given Bézier surface. A Bézier curve defined by the control points $\left\{\mathbf{r}_{0,j}^{m,n}, \ldots, \mathbf{r}_{m,j}^{m,n}\right\}$ in the *j*-th row of the given control net is denoted as $\mathbf{r}_{\square,j}^{m,n}(u) \equiv \sum_{i=0}^{m} \mathbf{r}_{i,j}^{m,n} \cdot B_i^m(u)$. Similarly, a curve defined by the control points $\left\{\mathbf{r}_{i,0}^{m,n}, \ldots, \mathbf{r}_{i,n}^{m,n}\right\}$ in the *i*-th column is denoted as $\mathbf{r}_{i,\square}^{m,n}(v) \equiv \sum_{j=0}^{n} \mathbf{r}_{i,j}^{m,n} \cdot B_j^m(v)$. Hence, Eqn. (24) can be expanded as follows:

$$\mathbf{r}^{m,n}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{r}_{i,j}^{m,n} \cdot B_i^m(u) \cdot B_j^n(v) = \sum_{j=0}^{n} \mathbf{r}_{\square,j}^{m,n}(u) \cdot B_j^n(v) = \sum_{i=0}^{m} \mathbf{r}_{i,\square}^{m,n}(v) \cdot B_i^m(u)\,. \tag{25}$$

The rib and the fan of the curve $\mathbf{r}_{\square,j}^{m,n}(u)$ are denoted as $\mathbf{r}_{\square,j}^{m-1,n}(u)$ and $\mathbf{p}_{\square,j}^{m-2,n}(u)$, respectively. According to Theorem 1, we can decompose the curve $\mathbf{r}_{\square,j}^{m,n}(u)$ as follows:

$$\mathbf{r}_{\square,j}^{m,n}(u) = \mathbf{r}_{\square,j}^{m-1,n}(u) + 2u(1-u)\mathbf{p}_{\square,j}^{m-2,n}(u). \tag{26}$$

Based on Eqn. (5), the control points of the rib $\mathbf{r}_{\square,j}^{m-1,n}(u)$ is expressed using the control points of the higher rib $\mathbf{r}_{\square,j}^{m,n}(u)$ as follows:

$$\mathbf{r}_{i,j}^{m-1,n} = \frac{1}{(m-1)}\left((m-1-i)\mathbf{r}_{i,j}^{m,n} + i\,\mathbf{r}_{i+1,j}^{m,n}\right) \quad \text{for } 0 \le i \le m-1. \tag{27}$$

Based on Eqn. (7), the control vectors of the fan $\mathbf{p}_{\square,j}^{m-2,n}(u)$ is expressed as follows:

$$\mathbf{p}_{i,j}^{m-2,n} = \mathbf{r}_{i+1,j}^{m,n} - \frac{1}{2}\left(\mathbf{r}_{i,j}^{m,n} + \mathbf{r}_{i+2,j}^{m,n}\right) \quad \text{for } 0 \le i \le m-2. \tag{28}$$

The rib and fan of the curve $\mathbf{r}_{i,\square}^{m,n}(v)$ is denoted as $\mathbf{r}_{i,\square}^{m,n-1}(v)$ and $\mathbf{q}_{i,\square}^{m,n-2}(v)$, respectively. Hence, the following decomposition is possible similarly with Eqn. (26):

$$\mathbf{r}_{i,\square}^{m,n}(v) = \mathbf{r}_{i,\square}^{m,n-1}(v) + 2v(1-v)\mathbf{q}_{i,\square}^{m,n-2}(v). \tag{29}$$

The control points and vectors of the rib and fan are defined similarly with Eqns. (27) and (28) as follows:

$$\mathbf{r}_{i,j}^{m,n-1} = \frac{1}{(n-1)}\left((n-1-j)\mathbf{r}_{i,j}^{m,n} + j\,\mathbf{r}_{i,j+1}^{m,n}\right) \quad \text{for } 0 \le j \le n-1 \tag{30}$$

$$\mathbf{q}_{i,j}^{m,n-2} = \mathbf{r}_{i,j+1}^{m,n} - \frac{1}{2}\left(\mathbf{r}_{i,j}^{m,n} + \mathbf{r}_{i,j+2}^{m,n}\right) \quad \text{for } 0 \le j \le n-2. \tag{31}$$

In the above, we considered a single transformation of control points along a certain parameter direction, either *u* or *v*. However, a successive transformation is possible to the previously transformed result regardless of parameter directions chosen. Specially, we are interested in following composite transformations that are necessary to derive the ribs and three types of fans of a tensor product Bézier surface:

1. $$\mathbf{r}^{m,n} \to \left\{\begin{array}{l} \xrightarrow[u]{\text{rib}} \mathbf{r}^{m-1,n} \xrightarrow[v]{\text{rib}} \\ \xrightarrow[v]{\text{rib}} \mathbf{r}^{m,n-1} \xrightarrow[u]{\text{rib}} \end{array}\right\} \to \mathbf{r}^{m-1,n-1} \tag{32}$$

2. $$\mathbf{r}^{m,n} \to \left\{\begin{array}{l} \xrightarrow[u]{\text{fan}} \mathbf{p}^{m-2,n} \xrightarrow[v]{\text{rib}} \\ \xrightarrow[v]{\text{rib}} \mathbf{r}^{m,n-1} \xrightarrow[u]{\text{fan}} \end{array}\right\} \to \mathbf{p}^{m-2,n-1} \tag{33}$$

3. $$\mathbf{r}^{m,n} \to \left\{\begin{array}{l} \xrightarrow[u]{\text{rib}} \mathbf{r}^{m-1,n} \xrightarrow[v]{\text{fan}} \\ \xrightarrow[v]{\text{fan}} \mathbf{q}^{m,n-2} \xrightarrow[u]{\text{rib}} \end{array}\right\} \to \mathbf{q}^{m-1,n-2} \tag{34}$$

4. $$\mathbf{r}^{m,n} \to \left\{\begin{array}{l} \xrightarrow[u]{\text{fan}} \mathbf{p}^{m-2,n} \xrightarrow[v]{\text{fan}} \\ \xrightarrow[v]{\text{fan}} \mathbf{q}^{m,n-2} \xrightarrow[u]{\text{fan}} \end{array}\right\} \to \mathbf{f}^{m-2,n-2} \tag{35}$$

where each arrow means a transformation of control points into those of lower degrees. The transformation type is shown above the arrow, and transformation direction below the arrow. Note that the sequence of transformations does not affect the result of a composite transformation. The first composite transformation defines a rib and the others define three types of fans of a given Bézier surface, which are u-directional, v-directional, and bidirectional fans. An example of the composite transformation in Eqn. (32) is applying Eqn. (30) to Eqn. (27). By combining Eqns. (28) and (30), Eqns. (27) and (31), and Eqns. (28) and (31), we can get examples of Eqns. (33), (34), and (35), respectively.

3.2.2. Rib and Its Control Points

For a given Bézier surface $\mathbf{b}(u,v) \equiv \mathbf{r}^{m,n}(u,v)$, we define the rib control points $\{\mathbf{r}_{i,j}^{k,l}\}$ of degree (k, l) by transforming the control points of the higher degree as follows:

$$\mathbf{r}_{i,j}^{k,l} = \frac{1}{k\,l}\Big((l-j)\big((k-i)\mathbf{r}_{i,j}^{k+1,l+1} + i\mathbf{r}_{i+1,j}^{k+1,l+1}\big) + j\big((k-i)\mathbf{r}_{i,j+1}^{k+1,l+1} + i\,\mathbf{r}_{i+1,j+1}^{k+1,l+1}\big)\Big) \text{ for } 1 \le k < n \text{ and } 1 \le l < m, \tag{36}$$

Note that, in the above definition, a single rib control point $\mathbf{r}_{i,j}^{k,l}$ is defined by four neighboring control points of the higher degree. This definition is the direct consequence of the transformation in Eqn. (32).

A rib $\mathbf{r}^{k,l}(u,v)$ of degree (k, l) is a Bézier surface of Eqn. (24) defined by the rib control points of the ancestor rib $\mathbf{r}^{k+1,l+1}(u,v)$ of degree (k, l) in Eqn. (36). Regardless of degrees k and l, the four corners of the rib are identical to those of the given surface: i.e., $\{\mathbf{r}_{0,0}^{k,l} = \mathbf{b}_{0,0}, \mathbf{r}_{k,0}^{k,l} = \mathbf{b}_{m,0}, \mathbf{r}_{0,n}^{k,l} = \mathbf{b}_{0,n}, \mathbf{r}_{k,l}^{k,l} = \mathbf{b}_{m,n}\}$. Hence, the lowest rib $\mathbf{r}^{1,1}(u,v)$ is a hyperbolic paraboloid (or doubly ruled surface) defined by above four control points.

3.2.3. Fan and Its Control Vectors

The results of the three transformations described in Eqns. (33), (34), and (35) corresponds to the control vectors of fans of Bézier Surface, which are expressed as follows:

$$\mathbf{p}_{i,j}^{k,l} = \frac{1}{l}\left((l-j)\left(\mathbf{r}_{i+1,j}^{k+2,l+1} - \frac{1}{2}\left(\mathbf{r}_{i,j}^{k+2,l+1} + \mathbf{r}_{i+2,j}^{k+2,l+1}\right)\right) + j\left(\mathbf{r}_{i+1,j+1}^{k+2,l+1} - \frac{1}{2}\left(\mathbf{r}_{i,j+1}^{k+2,l+1} + \mathbf{r}_{i+2,j+1}^{k+2,l+1}\right)\right)\right) \text{ for } 0 \le k \le n-2 \text{ and } 1 \le l \le m-1, \tag{37}$$

$$\mathbf{q}_{i,j}^{k,l} = \frac{1}{k}\left((k-i)\left(\mathbf{r}_{i,j+1}^{k+1,l+2} - \frac{1}{2}\left(\mathbf{r}_{i,j}^{k+1,l+2} + \mathbf{r}_{i,j+2}^{k+1,l+2}\right)\right) + i\left(\mathbf{r}_{i+1,j+1}^{k+1,l+2} - \frac{1}{2}\left(\mathbf{r}_{i+1,j}^{k+1,l+2} + \mathbf{r}_{i+1,j+2}^{k+1,l+2}\right)\right)\right) \text{ for } 1 \le k \le n-1 \text{ and } 0 \le l \le m-2, \tag{38}$$

$$\mathbf{f}_{i,j}^{k,l} = \left(\mathbf{r}_{i+1,j+1}^{k+2,l+2} - \frac{1}{2}\left(\mathbf{r}_{i,j+1}^{k+2,l+2} + \mathbf{r}_{i+2,j+1}^{k+2,l+2}\right)\right) - \frac{1}{2}\left(\left(\mathbf{r}_{i+1,j}^{k+2,l+2} - \frac{1}{2}\left(\mathbf{r}_{i,j}^{k+2,l+2} + \mathbf{r}_{i+2,j}^{k+2,l+2}\right)\right) + \left(\mathbf{r}_{i+1,j+2}^{k+2,l+2} - \frac{1}{2}\left(\mathbf{r}_{i,j+2}^{k+2,l+2} + \mathbf{r}_{i+2,j+2}^{k+2,l+2}\right)\right)\right), \text{ for } 0 \le k \le m\text{-}2 \text{ and } 0 \le l \le n\text{-}2. \tag{39}$$

Control vectors of Eqns. (37) and (38) is defined by six neighboring control points placed in a sub-mesh of size 3×2 or 2×3. Control vectors of Eqn. (39) is defined by nine neighboring control points placed in a sub-mesh of size 3×3. Using these control vectors, we define three types of fans: u-directional fan $\mathbf{p}^{k,l}(u,v)$, v-directional fan $\mathbf{q}^{k,l}(u,v)$, and bidirectional fan $\mathbf{f}^{k,l}(u,v)$, as vector fields of a tensor product form: i.e., $\mathbf{p}^{k,l}(u,v) \equiv \sum_{i=0}^{k}\sum_{j=0}^{l}\mathbf{p}_{i,j}^{k,l} \cdot B_i^k(u) \cdot B_j^l(v)$. For notational convenience, we define a composite fan to simplify the expression of the linear combination of three fans.

$$\boldsymbol{\varphi}^{k,l}(u,v) \equiv 2u(1-u)\,\mathbf{p}^{k,l+1}(u,v) + 2v(1-v)\,\mathbf{q}^{k+1,l}(u,v) + 4uv(1-u)(1-v)\,\mathbf{f}^{k,l}(u,v). \tag{40}$$

3.2.4. Theory

In this section, we describe how to decompose a Bézier surface into a rib and three fans using the transformations defined in Eqns. (36)-(39) for the control points and vectors.

Theorem 2. (Rib and Fan of Bézier Surface) A Bézier surface $\mathbf{b}(u,v) \equiv \mathbf{r}^{m,n}(u,v)$ of degree (m,n) for $m,n \ge 2$ is decomposed into a rib $\mathbf{r}^{m-1,n-1}(u,v)$ of degree $(m-1,n-1)$ and three fans, $\mathbf{p}^{m-2,n-1}(u,v)$, $\mathbf{q}^{m-1,n-2}(u,v)$, and $\mathbf{f}^{m-2,n-2}(u,v)$ as follows:

$$\mathbf{b}(u,v) \equiv \mathbf{r}^{m,n}(u,v) = \mathbf{r}^{m-1,n-1}(u,v) + 2u(1-u)\,\mathbf{p}^{m-2,n-1}(u,v) + 2v(1-v)\,\mathbf{q}^{m-1,n-2}(u,v) + 4uv(1-u)(1-v)\,\mathbf{f}^{m-2,n-2}(u,v)$$
$$= \mathbf{r}^{m-1,n-1}(u,v) + \boldsymbol{\varphi}^{m-2,n-2}(u,v) \quad (41)$$

Proof> We prove Eqn. (41) by the straightforward evaluation based on the definitions above and two consecutive decompositions of Bézier curve along *u* and *v* parameters using Theorem 1. First, for a fixed value of u, say u_0, the Bézier curve $\mathbf{r}^{m,n}(u_0,v)$ is decomposed as follows based on Eqns. (25) and (26):

$$\mathbf{r}^{m,n}(u_0,v) = \sum_{j=0}^{n} \mathbf{r}^{m,n}_{\square,j}(u_0)\cdot B^n_j(v) = \sum_{j=0}^{n}\left(\mathbf{r}^{m-1,n}_{\square,j}(u_0) + 2u_0(1-u_0)\mathbf{p}^{m-2,n}_{\square,j}(u_0)\right)\cdot B^n_j(v)$$
$$= \left(\sum_{j=0}^{n}\mathbf{r}^{m-1,n}_{\square,j}(u_0)\cdot B^n_j(v)\right) + \left(2u_0(1-u_0)\sum_{j=0}^{n}\mathbf{p}^{m-2,n}_{\square,j}(u_0)\cdot B^n_j(v)\right) = \mathbf{r}^{m-1,n}(u_0,v) + 2u_0(1-u_0)\mathbf{p}^{m-2,n}(u_0,v). \quad (42)$$

Since Eqn. (42) holds for any value of *u*, the following equation holds:

$$\mathbf{r}^{m,n}(u,v) = \mathbf{r}^{m-1,n}(u,v) + 2u(1-u)\mathbf{p}^{m-2,n}(u,v)\,. \quad (43)$$

Now, we fix the value of v as v_0, and evaluate the Eqn. (43) as follows:

$$\mathbf{r}^{m,n}(u,v_0) = \mathbf{r}^{m-1,n}(u,v_0) + 2u(1-u)\mathbf{p}^{m-2,n}(u,v_0) = \left(\sum_{i=0}^{m-1}\mathbf{r}^{m-1,n}_{i,\square}(v_0)\cdot B^{m-1}_i(u)\right) + 2u(1-u)\left(\sum_{i=0}^{m-2}\mathbf{p}^{m-2,n}_{i,\square}(v_0)\cdot B^{m-2}_i(u)\right) \quad (44)$$

where $\mathbf{p}^{m-2,n}_{i,\square}(v)$ denotes Bézier vector fields defined by the *i*-th column of control points of $\mathbf{p}^{m-2,n}(u,v)$. The term in the first parenthesis of Eqn. (44) is further evaluated as follows based on Eqn. (29):

$$\sum_{i=0}^{m-1}\mathbf{r}^{m-1,n}_{i,\square}(v_0)\cdot B^{m-1}_i(u) = \sum_{i=0}^{m-1}\left(\mathbf{r}^{m-1,n-1}_{i,\square}(v_0) + 2v_0(1-v_0)\mathbf{q}^{m-1,n-2}_{i,\square}(v_0)\right)\cdot B^{m-1}_i(u)$$
$$= \left(\sum_{i=0}^{m-1}\mathbf{r}^{m-1,n-1}_{i,\square}(v_0)\cdot B^{m-1}_i(u)\right) + 2v_0(1-v_0)\left(\sum_{i=0}^{m-1}\mathbf{q}^{m-1,n-2}_{i,\square}(v_0)\cdot B^{m-1}_i(u)\right) \quad (45)$$
$$= \mathbf{r}^{m-1,n-1}(u,v_0) + 2v_0(1-v_0)\mathbf{q}^{m-1,n-2}(u,v_0).$$

The term in the second parenthesis of Eqn. (44) is further decomposed as follows:

$$\sum_{i=0}^{m-2}\mathbf{p}^{m-2,n}_{i,\square}(v_0)\cdot B^{m-2}_i(u) = \sum_{i=0}^{m-2}\left(\mathbf{p}^{m-2,n-1}_{i,\square}(v_0) + 2v_0(1-v_0)\mathbf{f}^{m-2,n-2}_{i,\square}(v_0)\right)\cdot B^{m-1}_i(u)$$
$$= \left(\sum_{i=0}^{m-2}\mathbf{p}^{m-2,n-1}_{i,\square}(v_0)\cdot B^{m-1}_i(u)\right) + 2v_0(1-v_0)\left(\sum_{i=0}^{m-2}\mathbf{f}^{m-2,n-2}_{i,\square}(v_0)\cdot B^{m-1}_i(u)\right) \quad (46)$$
$$= \mathbf{p}^{m-2,n-1}(u,v_0) + 2v_0(1-v_0)\mathbf{f}^{m-2,n-2}(u,v_0)$$

By applying Eqns. (45) and (46) to Eqn. (44), and replacing v_0 by v, we can derive the desired Eqn. (41). **Q.E.D.**

Corollary 2. A Bézier surface $\mathbf{b}(u,v) \equiv \mathbf{r}^{m,n}(u,v)$ of degree (m,n) for $m,n \geq 2$ can be decomposed into a single rib of degree (*m-k, n-k*) and a sequence of *k* composite fans as follows:

$$\mathbf{r}^{m,n}(u,v) = \mathbf{r}^{m-k,n-k}(u,v) + \sum_{i=1}^{k}\boldsymbol{\varphi}^{m-1-k,n-1-k}(u,v) \quad \text{for } 1 \leq k \leq \mathrm{MIN}(m,n)-1\,. \quad (47)$$

As a direct consequence of Theorem 2, Corollary 2 states that a Bézier surface can be further decomposed into one rib of a lower degree and a sequence of composite fans. For example, a bi-cubic Bézier curve $\mathbf{b}(u,v) \equiv \mathbf{r}^{3,3}(u,v)$ (as in Fig. 5) can be decomposed as follows:

$$\mathbf{r}^{3,3}(u,v) = \mathbf{r}^{2,2}(u,v) + \boldsymbol{\varphi}^{1,1}(u,v) = \mathbf{r}^{1,1}(u,v) + \boldsymbol{\varphi}^{1,1}(u,v) + \boldsymbol{\varphi}^{0,0}(u,v) \quad (48)$$

4. EXAMPLES

In this section, we present geometric design examples based on the theories proposed in this paper. (The details are explained in the captions of each figure.) Fig. 3 and Fig. 4 show typical patterns of ribs and fans for Bézier curves of degrees are 9 and 10. Fig. 5 shows an example of ribs and fans of a simple Bézier surface of degree (3,3). (More examples can be found in the web site[6].)

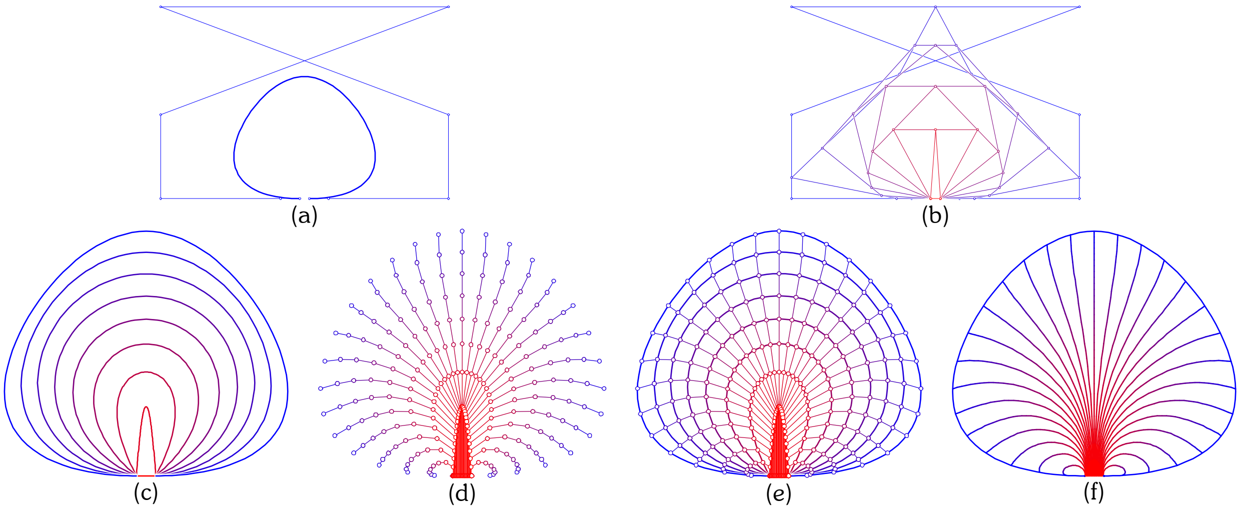


Fig. 3. A Bézier curve of degree 9 with the shape of a hand-held cooling fan: (a) given curve $\mathbf{b}(t) \equiv \mathbf{r}^9(t)$ and its control points; (b) the control points of all the lower ribs whose color becomes red as its degree decreases; (c) a sequence of ribs defined by rib control points of (b); (d) fan lines generated by connecting sampled fan vectors of a sequence of scaled fans; (e) the overlay of (c) and (d); and (f) fan curves whose control points are defined by the fan lines of (d).

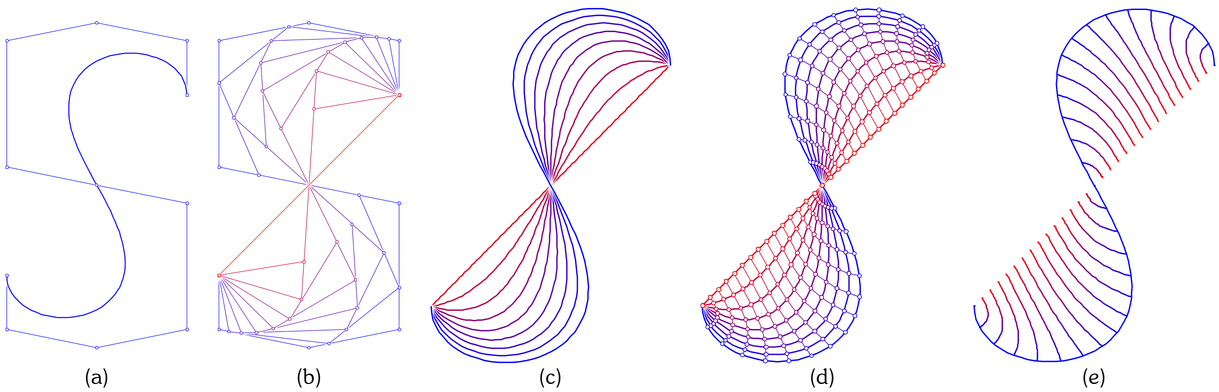
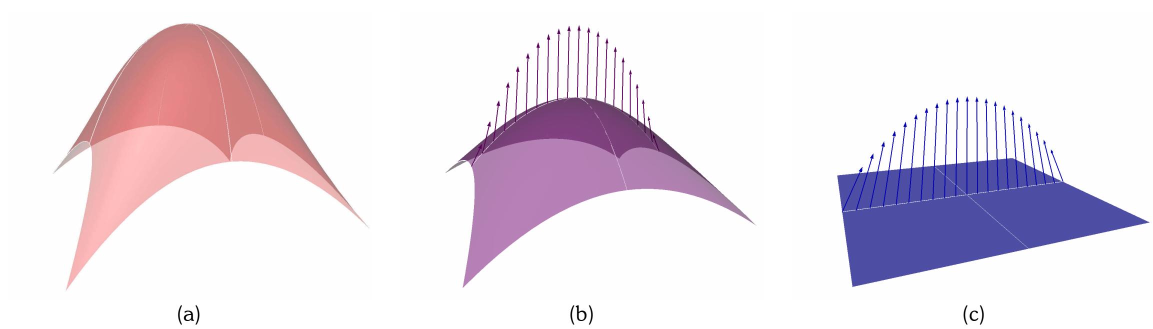


Fig. 4. A Bézier curve $\mathbf{b}(t) = \mathbf{r}^{10}(t)$ with the shape of the alphabet S. In this example, the two lowest ribs coincide due to the symmetry of the shape. Hence, in (c), there appear 9 ribs rather than 10: (a) the given curve and its control points; (b) rib control points; (c) ribs; (d) ribs and fan lines; and (e) fan curves.



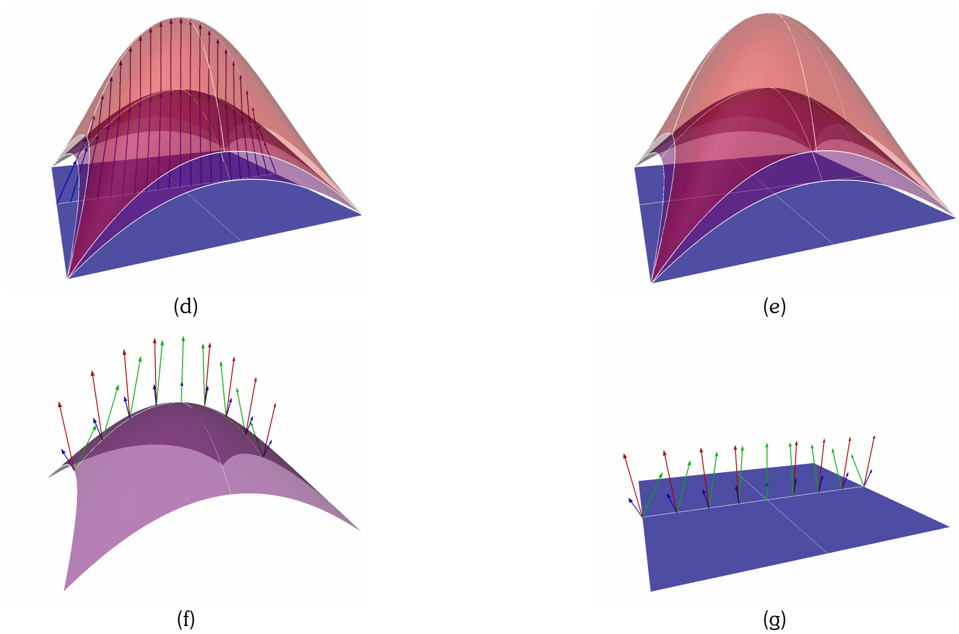


Fig. 5. A bi-cubic Bézier surface $\mathbf{b}(u,v)=\mathbf{r}^{3,3}(u,v)$: (a) the shape of the given surface. White curves denote iso-curves at u=0.5 and v=0.5; (b) the rib $\mathbf{r}^{2,2}(u,v)$ and sampled vectors of the composite fan $\boldsymbol{\varphi}^{1,1}(u,0.5)$; (c) the rib $\mathbf{r}^{1,1}(u,v)$ and sampled vectors of the composite fan $\boldsymbol{\varphi}^{0,0}(u,0.5)$; (d) overlay of (a), (b), and (c) including ribs and fans; (e) overlay of (a), (b), and (c) without fans; (f) the rib $\mathbf{r}^{2,2}(u,v)$ and sampled vectors of three fans $\mathbf{p}^{1,2}(u,0.5)$, $\mathbf{q}^{2,1}(u,0.5)$, and $\mathbf{f}^{1,1}(u,0.5)$ in red, green, and blue arrows, respectively; and (g) the rib $\mathbf{r}^{1,1}(u,v)$ and sampled vectors of $\mathbf{p}^{0,1}(u,0.5)$, $\mathbf{q}^{1,0}(u,0.5)$, and $\mathbf{f}^{0,0}(u,0.5)$.

5. CONCLUDING REMARKS

We have presented new geometric concepts of Bézier curves and surfaces: ribs and fans. We have shown that a Bézier curve of degree *n* is decomposed into a rib of degree (*n*-1) and a fan of degree (*n*-2), and a Bézier surface of degree (*m*, *n*) into a rib of degree (*m*-1, *n*-1) and three fans of degrees (*m*-1, *n*-2), (*m*-2, *n*-1), and (*m*-2, *n*-2), respectively. We have also presented the methods to transform the control points of a given Bézier curve or surface into its rib control points and fan control vectors of lower degrees.

Using the ribs and fans of Bézier curves and surfaces, we can design interesting geometric patterns resembling the features of the given curves and surfaces. The presented methods can be used in design applications that require natural and aesthetic patterns such as flowers, leaves, and sea shells[6].

As ribs and fans are globally defined over given Bézier control points, there is some limitation to apply the proposed method directly to a piecewise curve or surface. When we apply the decomposition to each segment of the piecewise curve or surface, it may result in unwanted, discontinuous piecewise ribs and fans, which are unavoidable with the current methods presented in this paper. Consideration of ribs and fans for a B-spline curve or surface will be an interesting further research as well as rational extensions.

6. REFERENCES

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