Entropy Assessment of the Symmetry in Tessellation Designs Based on Information Theory

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Abstract. The tessellation of an architectural facade is a regular and aesthetic symmetrical two-dimensional pattern that is generated by a unit pattern based on a symmetry group. The design information that utilizes symmetry in the design process will be delivered to the construction stage. Based on information theory, this study attempts to quantify the entropy of symmetry design, which will serve as a standardized and repetitive assessment indicator of the symmetry design patterns for a designer. This will be delivered to the subsequent construction stage as procedural information to facilitate an understanding of the constructability of the design. This article quantifies the design information for 17 types of two-dimensional symmetry patterns to illustrate that the amount of entropy will reflect the complexity of the symmetry patterns, which, in part, facilitate design and construction.

Keywords: Symmetry pattern, Quantification, Information theory, Entropy
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1 INTRODUCTION

The tessellation design of a building facade is an important work of architecture; it utilizes the simple unit and particular symmetrical rules to form a symmetrical pattern that is regular and aesthetic, as shown in Figure 1. Group theory is a mathematical tool for studying symmetrical patterns. Using a symmetry group to analyze a pattern can generate a variety of different symmetrical patterns by finite unit pattern.

These symmetrical rules should be delivered to the constructor as procedural information to improve design efficiency and the constructability in the subsequent construction stage. In the design stage, an architect can judge the constructability through this procedural information. In the construction stage, if the constructor needs more procedural information, which is delivered from the design stage, this indicates that the design is difficult to construct. In contrast, if the constructor needs less information, this indicates that this design is relatively simple to construct. The key is uncertainty. That is, when uncertainty is high, it is relatively difficult for construction for constructor, and vice versa. Therefore, a quantification of the uncertainty that is procedural information in the design stage can be used to measure the complexity of the pattern design and the constructability of the design in the construction stage [1].

Figure 1: The symmetry patterns in the tessellation of an architectural facade design.

In previous studies, many researchers have utilized a symmetry group to study symmetric patterns. In the book "Symmetries of Islamic Geometrical Patterns", Syed Jan Aba and Amer Shaker Salman systematically discussed the geometric design of Islamic architecture that utilizes a symmetry group [4]. Yi utilized symmetry group theory to analyze 17 different patterns in the design of a building facade [12]. Aziz Khamjane proposed a method to generate Islamic star patterns based on symmetry group theory [5]. José Pedro Sousa utilized digital technology to further explore the application of symmetry in architectural design and manufacturing in the context of teaching experiments [9]. With the promotion of information technology, increasingly more research has applied computers to the generation of symmetric patterns. Based on previous studies, this study utilizes the communication model of information theory to quantify the entropy of procedural information in the hierarchical decision-making of the generation process of symmetrical patterns. Accordingly, this paper further discusses the influence of quantification on the subsequent design and construction, which will serve as a standardized and repetitive assessment indicator of symmetry pattern design for architects to facilitate an understanding of the constructability of design.

2 MAIN IDEA

2.1 Symmetry Group and Tessellation

Mathematicians call the collection of all the symmetry operations or motions that leave a particular geometric object fixed its symmetry group. These identity transformations include five basic motions: translation, mirror, rotation, sliding mirror and identity transformation [7]. For example, the symmetry group of a square is the following set:

$$G = \{ I, R_{\pi/2}, R_{\pi}, R_{3\pi/2}, M_a, M_b, M_c, M_d \} \tag{2.1}$$

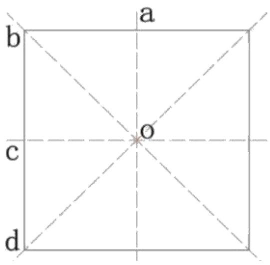

Figure 2: The symmetry in a square.

where I represents an invariant transformation with on effect, that is, a rotation of 360° or 0°. Rп/2, Rп and R3п/2 represent the rotation of 90°, 180° and 270°, respectively. Ma, Mb, Mc and Md represent the mirror images with a, b, c and d axes, respectively, as shown in Figure 2.

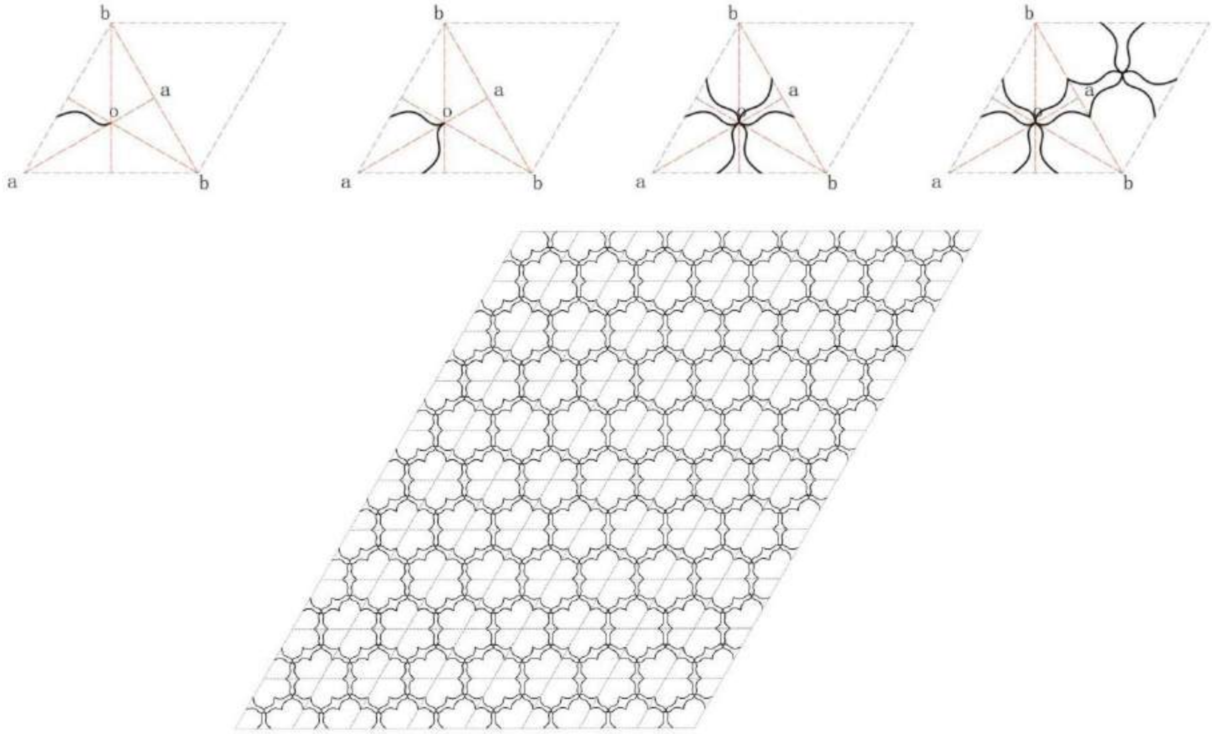

Figure 3: The generation of the unit motif of the “begonia” pattern.

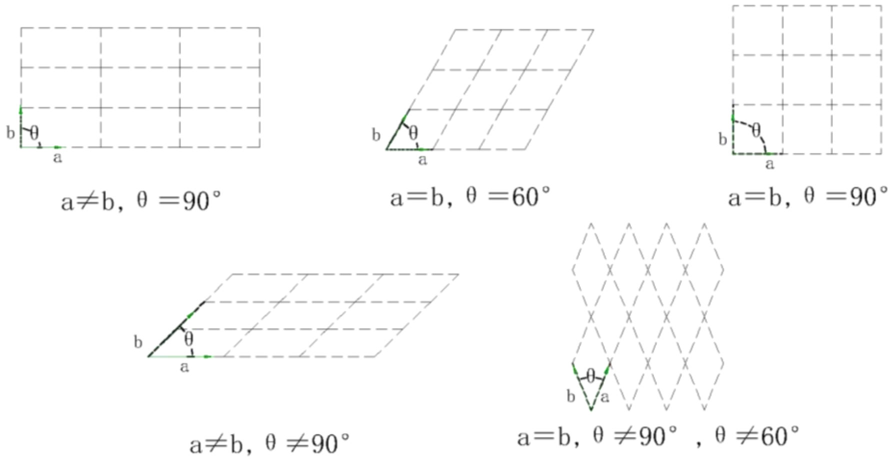

Figure 4: Five types of grid systems: (a) Rectangle, (b) Diamond with 60° angle, (c) Square, (d) Oblique parallelogram, and (e) Diamond without 60° and 90° angle.

According to Syed Jan Abas, a two-dimensional repeating pattern is generated by a unit motif along a grid system that is generated by two sets of parallel lines [4]. From left to right, the upper diagram of Figure 3 describes that the template motif within two sets of parallel lines, that is, a unit cell, generates the basic pattern unit motif within the dash line, according to the five types of transformation mentioned above. The different unit cell should form different grid systems, which become the grid of displacement symmetry behind the two-dimensional repeating pattern. Then, a unit motif will be copied into a complex repeating pattern along the grid system, as shown in Figure 3. According to Syed Jan Abas, there are five types of grid systems, which are oblique parallelograms with unequal edges, rectangles, diamonds without 60° angles, diamonds with 60° angles and a square, as shown in Figure 4 [4]. Different grid systems have their own symmetrical axis and central symmetry points.

2.2 The Judgement and Decision Level of a Two-dimensional Symmetrical Pattern

Based on the five different parallelogram grid systems mentioned above, we can further explore the analysis of possibility of internal symmetry by using the four equidistant transformations on the plane other than the identity transformation. According to the five grid systems, the pattern can be classified by whether this pattern has mirror symmetry, rotation symmetry and sliding mirror, etc. in the different situations of each level. In fact, architects also make design decisions based on whether the pattern has mirror symmetry, rotation symmetry, etc. Accordingly, the process of judgement in different levels is similar to the process of design decision-making by architects. The symmetrical pattern named p6m is taken as an example for the judgement of symmetrical classification, as shown in Figure 5. The p6m pattern is the most common symmetrical type of two-dimensional pattern. This representation of a symmetrical pattern is represented by the Hermann-Mauguin notation [3].

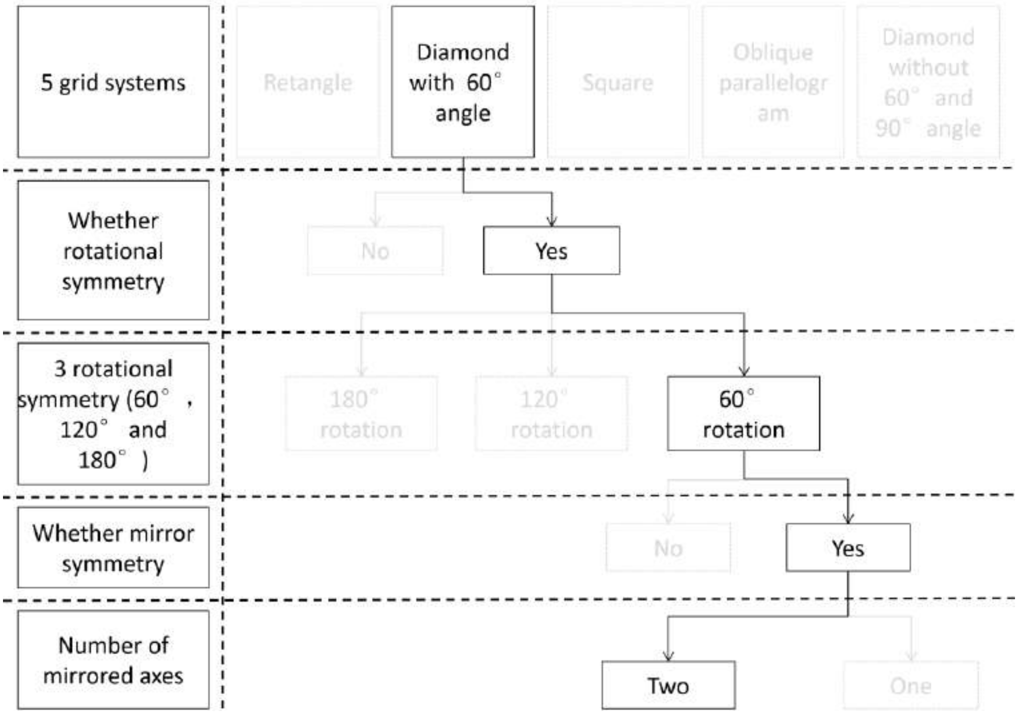

Figure 5: The analysis of different situations at each level of the p6m pattern.

2.3 Communication Model and Procedural Information

Claude Elwood Shannon proposed a mathematical communication model to measure the amount of information in a communication system, which became the basis of information theory [8]. The process of architecture design communication can also be explained by a flow chart, as shown in Figure 6. An architect uses drawings, models and sketches to express the design concept, that is, the process of encoding the information of a design concept into messages. The constructor decodes the received message into the design concept [10]. The same is true for the tessellation design. The design is encoded in symmetrical patterns by an architect who utilizes symmetry and is then delivered to the constructor. The constructor decodes the pattern for construction.

In this process, there is a wealth of procedural information, including geometric information and non-geometric information [2]. This procedural information should be recorded and delivered to the construction stage as an important part of design to improve the design quality. Therefore, we should quantify the information encoded. These quantified results should serve as an assessment of the complexity of the tessellation design and as the assessment of constructability for the subsequent construction stage.

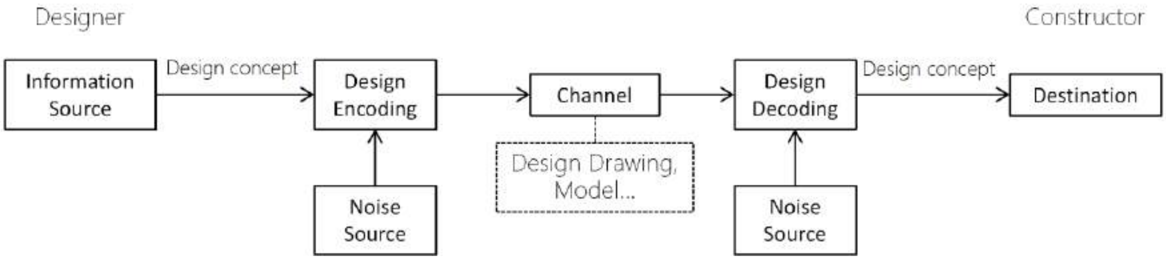

Figure 6: The mathematical model of the communication between the designer and constructor.

2.4 Quantification and entropy

To quantify the information, Shannon further suggested that information can be measured by calculating the uncertainty, or the unlikelihood, of the recognized situations with a logarithm function over probability [8].

$$H(X) = \sum p(X_i) log \frac{1}{p(X_i)} \quad (2.2)$$

where, $p(X_i)$ depends on all the possibilities that can be selected in the event, H(X) indicates the entropy of an event distribution X.

$$p(X_i) = \frac{X_i}{N} \quad (2.3)$$

Here, X_i indicates decision and N is the total number of all possible decisions.

As mentioned above, these 17 symmetrical types are generated based on the five different types of grid system and four different equidistant transformations [4]. For example, the p6m pattern is used to quantify the amount of information for each step of the symmetry type in Figure 5:

- Step 1: select one of the five grid systems. There are five possibilities in these events. According to formula 1, the entropy of this level can be qualified as: *1/5*log5*;
- Step 2: According to different grid systems, there are two possibilities in the selection of whether the design pattern has rotation symmetry at this level: *1/2*log2*;

- Step 3: if it has rotation symmetry, there are three possibilities for rotation that include 180° rotation, 120° rotation and 60° rotation. The entropy of this level can be qualified as *1/3*log3*; and
- Step 4: if the architect selects the 60° rotation, there are two possibilities regarding whether or not the design pattern has mirror symmetry at this level: *1/2*log2*.

Therefore, when the architect utilizes the four basic motions on five types of grid system to form the symmetrical pattern of this type named p6m, the entropy of the p6m pattern is calculated as follows:

$$H = \frac{1}{5}\log 5 + \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{2}\log 2 = 0.5997$$

Type of symmetrical pattern	*Grid system*	*Amount of entropy*
P1	Oblique parallelograms with unequal edges	*1/5*log5+1/2*log2 =0.2903*
p2		*1/5*log5+1/2*log2 =0.2903*
p1m	Rectangle	*1/5*log5+1/2*log2+1/2*log2+1/2*log2 =0.5913*
p1g		*1/5*log5+1/2*log2+1/2*log2+1/2*log2 =0.5913*
p2mm		*1/5*log5+1/2*log2+1/2*log2+1/2*log2 =0.5913*
p2mg		*1/5*log5+1/2*log2+1/2*log2+1/2*log2 =0.5913*
p2gg		*1/5*log5+1/2*log2+1/2*log2+1/2*log2 =0.5913*
c1m	Diamond without 60° and 90° angle	*1/5*log5+1/2*log2+1/2*log2 =0.4408*
c2mm		*1/5*log5+1/2*log2+1/2*log2 =0.4408*
p4	Square	*1/5*log5+1/2*log2+1/2*log2+1/2*log2 =0.5913*
p4mm		*1/5*log5+1/2*log2+1/2*log2+1/2*log2+1/2*log2 =0.7418*
p4gm		*1/5*log5+1/2*log2+1/2*log2+1/2*log2+1/2*log2 =0.7418*
p3	Diamond with 60° angle	*1/5*log5+1/2*log2+1/3*log3+1/2*log2=0.5998*
p3m1		*1/5*log5+1/2*log2+1/3*log3+1/2*log2++1/2*log2 +1/2*log2 =0.9008*
p31m		*1/5*log5+1/2*log2+1/3*log3+1/2*log2++1/2*log2 =0.7503*
p6		*1/5*log5+1/2*log2+1/3*log3+1/2*log2 =0.5998*
p6mm		*1/5*log5+1/2*log2+1/3*log3+1/2*log2 =0.5998*

Table 1: The quantification of 17 types of symmetrical patterns.

3 DEMONSTRATION

The tessellation of the architecture façade in South Fujian China is designed according to two-dimensional symmetrical patterns. The generation and quantification of four symmetrical patterns are described in this article as examples that demonstrate how the quantification of entropy facilitates an understanding of the constructability of design. The four symmetrical patterns are named "tortoise shell", "continuous 卐", "begonia" and "diamond", as shown in **Figure 7** .

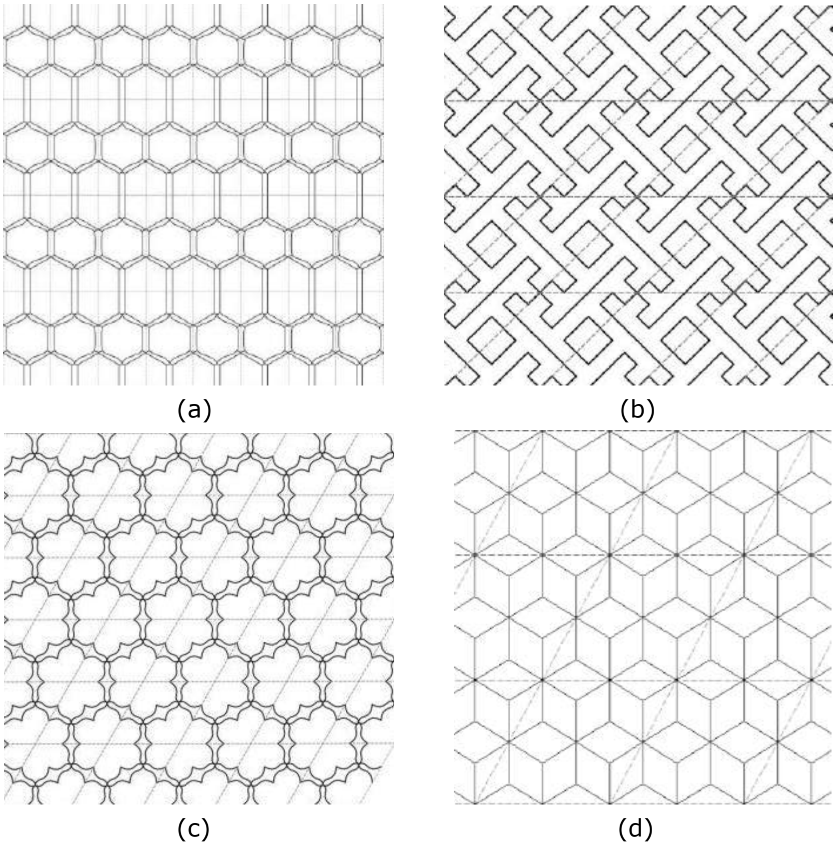

Figure 7: Four types of symmetrical patterns in the tessellation of the facade in the traditional architecture design in South Fujian, China: (a) Tortoise shell; (b) Continuous 卐; (c) Begonia; and (d) Diamond.

3.1 Quantification and Generation of Four Symmetrical Patterns

- The "tortoise shell" is generated along the rectangular grid system, which is one of the five grid systems. The generated unit motif by the template motif is mirror-symmetry with an a-a axis and has rotation-symmetry with point O, as shown in Figure 8. Therefore, as shown in Figure 9, there is rotational-symmetry and mirror-symmetry, respectively, at the respective decision levels. However, there is no sliding mirror symmetry. The entropy is quantified as *1/5*log5+1/2*log2+1/2*log2+1/2*log2=0.5913*.

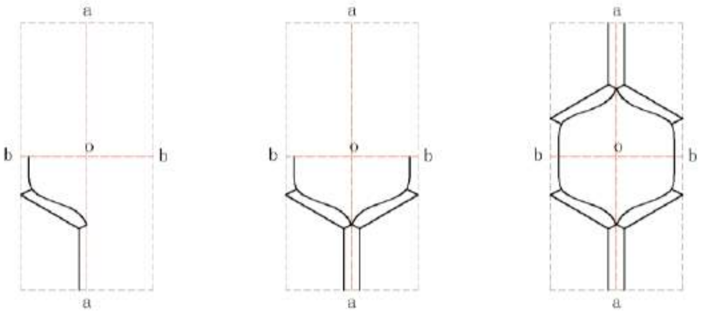

Figure 8: The generation of the unit motif of "tortoise shell".

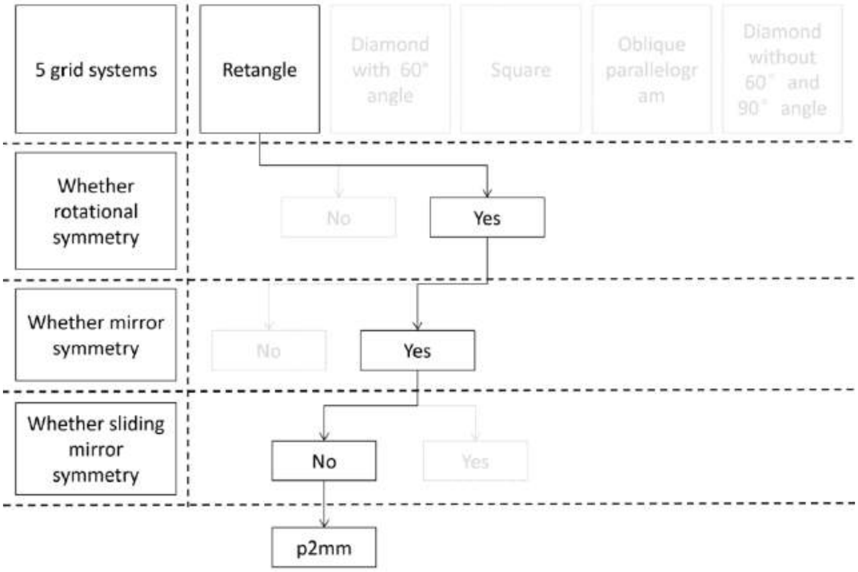

Figure 9: The analysis of different situations in each level of the p2mm pattern.

- The grid system of the "continuous 卐" pattern is oblique parallelograms with unequal edges. The unit motif is rotation-symmetry with point O, as shown in Figure 10. Therefore, the entropy is quantified according to Figure 11: *1/5*log5+1/2*log2=0.2903*.
- According to Figure 3 mentioned above, the entropy of the "begonia" pattern is quantified as *1/5*log5+1/2*log2+1/3*log3+1/2*log2=0.5998*.
- The generation of the "diamond" pattern is consistent with "begonia", as shown in Figure 12. According to Figure 3, first, it takes the diamond with 60° as the grid system; second, it is judged regarding whether it has rotation-symmetry. Then, the pattern has rotation-symmetry with 60°; finally, the pattern has mirror-symmetry. Therefore, the entropy is quantified as *1/5*log5+1/2*log2+1/3*log3+1/2*log2=0.5998*.

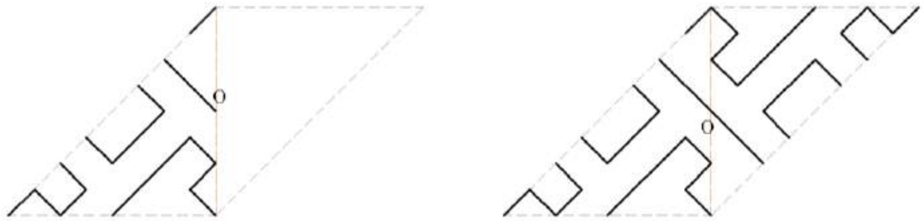

Figure 10: The generation of the unit motif of "continuous 卐".

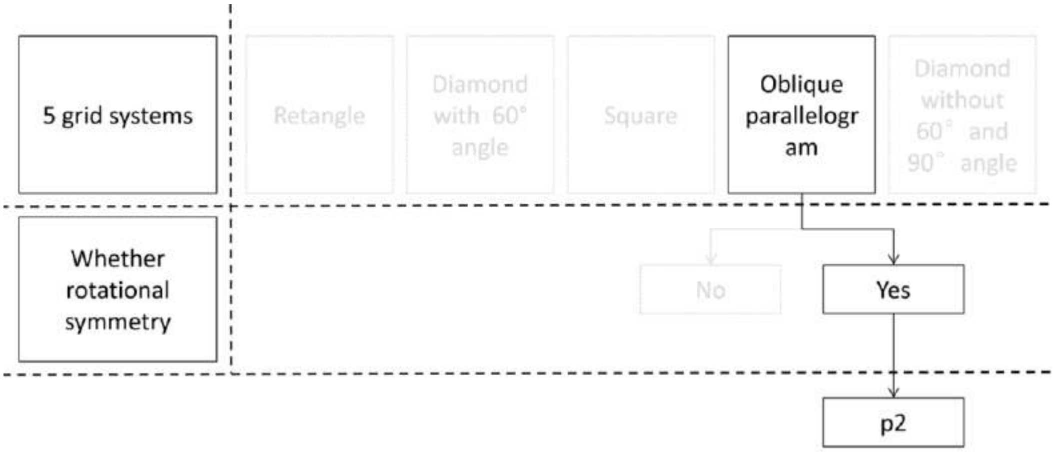

Figure 11: The analysis of different situations in each level of the p2 pattern.

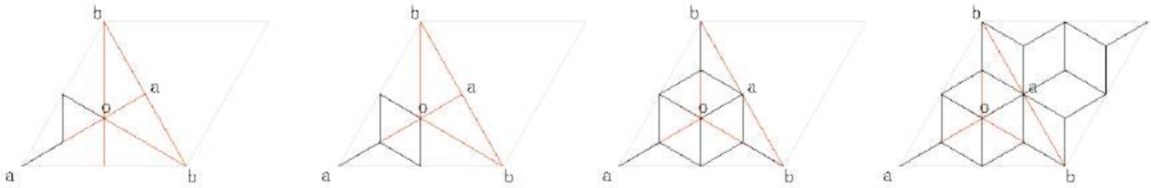

Figure 12: The generation of the unit motif of "diamond".

Symmetrical pattern	*Grid system*	*Type*	*Entropy*
Tortoise shell	Rectangle	p2mm	0.5913
Continuous 卐	Oblique parallelogram	p2	0.2903
Begonia	Diamond with 60°	p6mm	0.5998
Diamond	Diamond with 60°	P6mm	0.5998

Table 2: The quantified entropy of four symmetrical patterns.

As shown in Table 2, when the architect makes decisions on the “continuous 卐” pattern design, the entropy is the least in the four examples; the entropy of the “tortoise shell” pattern takes second place, and the entropy of the “begonia” and the “diamond” patterns are the highest. Although the “continuous 卐” pattern seems to be more complicated, the number of decision levels is relatively less than the other three types of symmetrical patterns. Thus, the cost is lower and the entropy in the decision-making process is less. The entropy of the “begonia” pattern and the “diamond” pattern is consistent. This is because the number of decision levels is the same, although the symmetrical patterns are different. Therefore, in Table 1, different symmetrical patterns might have the same entropy in their decision-making process. At the early design stage, an architect only needs to refer to the pattern type and its corresponding information entropy in Table 1. The amount of information entropy quantified will be used as a numerical indicator in design process to assist architect to understand the uncertainty of design, even the constructability in the subsequent construction stage.

3.2 The procedural information

To illustrate the role of entropy as procedural information in the design process, this study utilizes Rhino+Grasshopper to implement the generation of a symmetrical pattern by an architect. Rhino+Grasshopper is a visual programming software that is more consistent with an architect`s operating habits. Figure 13 shows the generation programs of the four symmetrical patterns mentioned above, which simulates the generation process. The two-dimensional symmetrical pattern has more possibilities for design when the template motif of the pattern is changed. Figure 14 shows that the tessellation is generated based on the p2 symmetrical pattern. When an architect makes appropriate modifications to the template motif, the generation of symmetrical patterns can make a variety of changes, which greatly improves the design efficiency.

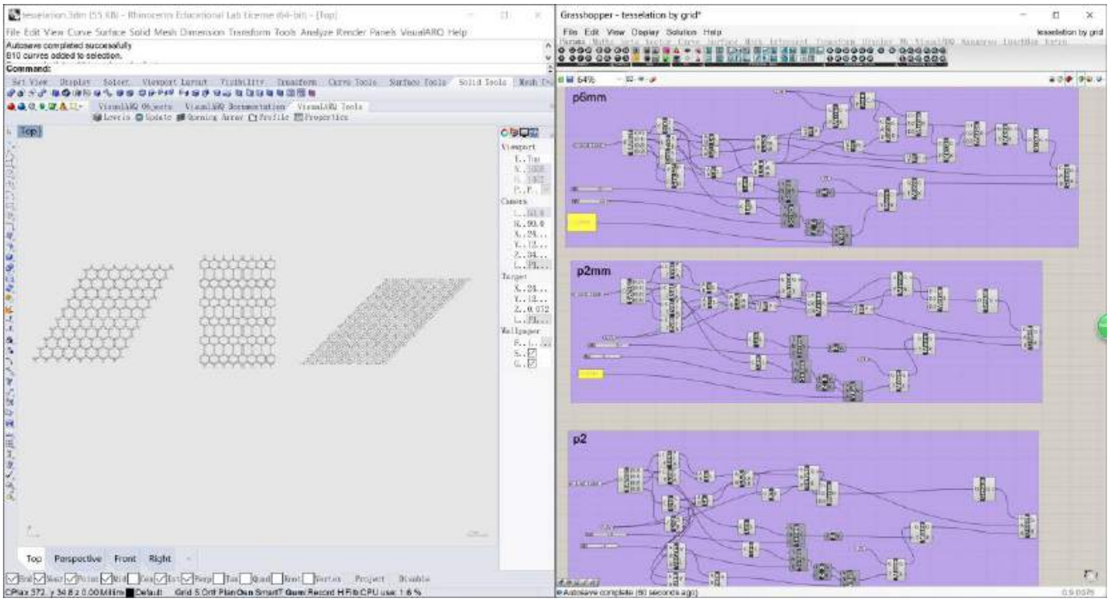

Figure 13: The generation programs of symmetrical patterns by Rhino+Grasshopper.

In fact, the entropy of the decision on the changed patterns by an architect is the same. That is, this consistency allows an architect to only consider the design of the template motif with the assistance of a computer. In the process of gradually generating a unit motif by the template motif, the information at various decision levels should be integrated into the design model as procedural information. In fact, generative modeling technology plays an important role in the design process: It takes procedural information as its main subject; it can also integrate specialized information in the modeling process, ensuring the particularity of design [11]. Thus, the quantified entropy should be integrated into the design model as procedural information. With the development of the design and construction stage, this procedural information will be delivered to the constructor along with the design model. Therefore, the procedural information and quantification of the different types of tessellation design and generation will provide more reference for the constructor.

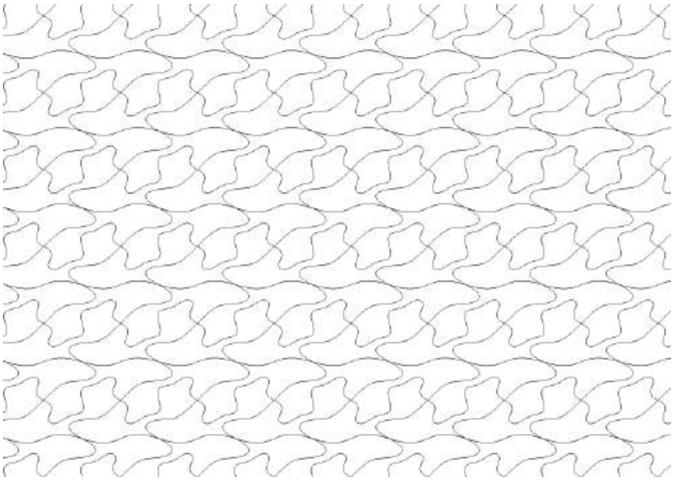

Figure 14: The symmetrical pattern generated based on the p2 pattern.

4 CONCLUSION

Tessellation design is a common method used in architectural facade design. A variety of different symmetry patterns are generated from a simple confined unit pattern, which is based on an analysis of patterns by symmetry group. Based on information theory, we should quantify the entropy of symmetry in the design to evaluate the amount of information of different symmetry patterns in the design process. The amount of information should serve as a reference indicator for a designer to evaluate the complexity of two-dimension symmetry pattern design. This will guide the subsequent construction. The entropy formulation therefore provides an information theory-based methodology for the quantitative assessment of tessellation design complexity, although, the methodology is currently only able to assess the complexity of constructability for repetitive and standardized elements.

In fact, the manufacturability of pattern design not only depends on the complexity of the symmetrical operators but also the shape of unit motif. There are lots of different types of unit motif in traditional tessellation design of facade in China. A comprehensive theory of design based on group theory proposed by Michael Leyton may be taken as a basic theory to discussed in the future work [6].

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REFERENCES

[1] Chang, M.-C.; Shih, S.-G.; Schmitt, G: Information theory-based approach for constructability assessment in truss structural systems, Automation in Construction, 82, 2017, 84–102. https://doi.org/10.1016/j.autcon.2017.06.010

[2] Eastman, C.; Teicholz, P.; Sacks, R.; Liston, K.: BIM Handbook: A Guide to Building Information Modeling for Owners, Managers, Designers, Engineers, and Contractors (Second), Hoboken, New Jersey: John Wiley & Sons, Inc, 2011.

[3] Hermann–Mauguin notation: https://en.wikipedia.org/wiki/Hermann%E2%80%93Mauguin_notation, Wikipedia.

[4] Jan, A. S.; Shaker, S. A.: Symmetries of Islamic Geometrical Patterns, London, World Scientific Publishing Co. Pte. Ltd, 1995.

[5] Khamjane, A.; Benslimane, R.: A computerized method for generating Islamic star patterns, Computer Aided Design, 97, 2018, 15–26. https://doi.org/10.1016/j.cad.2017.11.002

[6] Leyton, M.: A Generative Theory of Shape, Springer Science & Business Media, 2001.

[7] Mitra, N. J.; Pauly, M.; Wand, M.; Ceylan, D.: Symmetry in 3D geometry: Extraction and applications, Computer Graphics Forum, 32(6), 2013, 1–23. https://doi.org/10.1111/cgf.12010

[8] Shannon, C.E.: The Mathematical Theory of Communication, University of Illinois Press, 1949. http://doi.org/10.1002/j.1538-7305.1948.tb01338.x

[9] Sousa, J. P.; Xavier, J. P.: Symmetry-based generative design and fabrication: A teaching experiment, Automation in Construction, 51(C), 2015, 113–123. https://doi.org/10.1016/j.autcon.2014.11.001

[10] Wu, N.; Wang, M.: The integration of procedural information in traditional architectural design, Computer-Aided Design and Applications, 16(1), 2019, 13–24.https://doi.org/10.14733/cadaps.2019.13-24

[11] Wu, N.; Shih, S.-G.: A BIM Inspired Supporting Platform for Architectural Design, Computer-Aided Design and Applications, 12(3), 2015, 327–337. http://dx.doi.org/10.1080/16864360.2014.981463

[12] Yi, Z.; Yuan, C.: Building façade designs based on group theory, Interior Architecture of China, 6, 2012, 158–159.