





## Vector Construction Techniques for Circle Packing

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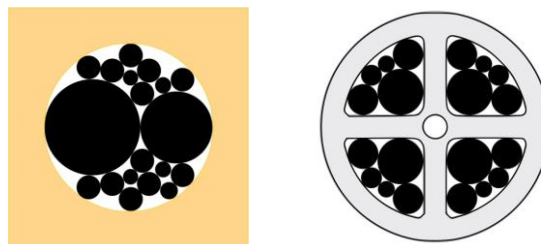
**Abstract.** This paper presents a method of vector-based constructions for circles and lines, which is presented in the context of a circle packing problem where it gives parametric descriptions of paths of centers of tangent circles and tangent vectors to those paths. The method is extended to common CAD constructions where it gives compact solutions that are easily coded in, and take advantage of, an object-oriented language.

**Keywords:** Circle Packing, Geometric Construction, CAD

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### 1 INTRODUCTION – CONSTRUCTION CONTEXT

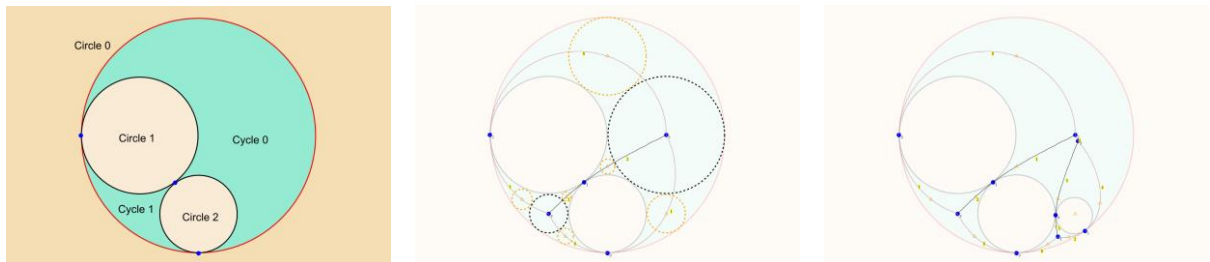
Design problems such as routing wires or pipes through a wall, Figure 1(a), and routing wires through a slip ring core, Figure 1(b), both reduce to the problem of packing a given set of circles into a geometric shape while satisfying some minimization criterion. With region boundaries involving circles, line segments and circular arcs, the packing involves fitting a given circle tangent to two lines, tangent to a circle and a line, and tangent to two circles. [3] does the packing in Figure 1(a) by finding a large enough enclosing circle then iteratively decreasing its radius and rearranging the packing. [1] treats packings in circles and other simple shapes as a global optimization problem and discusses industrial applications. [2] surveys the literature for these and related packings.



**Figure 1:** Routing wires: (a) through hole (b) through slip ring core.

The authors are developing test software to compare various packing algorithms and identify adjustments that would improve the packing. The software needs the center path arcs and the tangent vectors to the arcs as a function of the radius of the circle being fitted. The circle fittings are old problems that are solved for CAD systems by solving systems of equations [4],[5]. These solutions could be used, but managing the cases is awkward and getting the tangent vector is even more awkward. The authors have developed compact and visually intuitive methods for the fittings and for generating the center arcs and tangent vectors. Although the derivations are tedious, the resulting formulas are simple and the method can be generalized to other CAD line and circle constructions and offers advantages over the traditional methods [4],[5] by being intuitive and simple to code and test.

The challenge in packing a set of circles is knowing where to place the next one. In Figure 2(a), the two blue regions are the possible placement locations for a new circle. The question is: which region and where in that region?



**Figure 2:** Partial packings in a circle: (a) Blue regions show potential locations for next circle, (b) Interception graph for Figure 2(a), showing possible locations for center of next circle, and (c) Circle added to Figure 2(a) and updated interception graph.

In the authors' software, two graphs are used to analyze the placement. The first graph, the cycle graph, manages the open space in the packing. Referring to Figure 2(a), Circle #0 is the enclosing circle and Circle #1 and Circle #2 are the most recently placed circles. The nodes of the cycle graph are the circles and the edges are the points of tangency. The two cycles, traced counterclockwise, are Cycle #0,  $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ , and Cycle #1,  $0 \rightarrow 2 \rightarrow 1 \rightarrow 0$ .

The suitability of a particular cycle for placing a new circle depends on the size of the circle. Each cycle contains a second graph, the interception graph, which records the available space inside the cycle. Figure 2(b) shows the interception graphs for the two cycles in Figure 2(a). For each interception graph, the edges are the arcs shown, which are the locations of the centers of circles that are tangent to the two circles that lie on either side of the arc. If one of the circles is the enclosing circle, the arc is elliptic. If neither circle is the enclosing circle, then the arc is hyperbolic. Both arcs are parameterized by the radius of the circle being fitted. In the simplest case, the arcs emerge from the tangent points and meet at an interception point, which is the center of the largest circle that can be fitted in that part of the region (the dashed black circles in Figure 2(b)). The nodes of the interception graph are the tangent points, the interception points, and an additional type of point called a joining point (that joins pieces of arcs). (Visually, the interception graph is a special case of the circle Voronoi diagram of [3].) When a new circle is added, the corresponding interception graph must be modified and often the modifications are localized, which means that a complex interception network is built incrementally. If the radius is small enough, the modifications are restricted to the arc on which the center is located. The dashed orange circles in Figure 2(b) are the largest circles that can be placed on the arcs without modifying the graph beyond the arcs. Figure 2(c) shows the updated interception graph after a third circle is added and suggests the increasing complexity as more circles are added. The graph updating requires the tangent vector to the arc at the center's location.

The first part of this paper focuses on derivations of the tangency cases for packing and the second part focuses on applications of the method to some common CAD constructions.

## 2 NOTATION, CONVENTIONS AND PRELIMINARY COMMENTS

Points and general vectors will be represented by a bold uppercase letter,  $\mathbf{P}$ . For interactive usage in CAD constructions, a point,  $\mathbf{P}_{pick}$ , picked from the graphics display by the user, selects cases and subcases. A unit vector is represented as  $\hat{\mathbf{v}} = (v_x, v_y)$ , and its orthogonal complement, as  $\hat{\mathbf{v}}^\perp = (-v_y, v_x)$ . A unit vector from point  $\mathbf{Q}$  to point  $\mathbf{P}$  is  $\hat{\mathbf{v}} = (\mathbf{P} - \mathbf{Q}) / \|\mathbf{P} - \mathbf{Q}\|$ , where  $\|\mathbf{P} - \mathbf{Q}\|$  is the distance between  $\mathbf{Q}$  and  $\mathbf{P}$ .

Lines  $Ax + By = C$  are in unit-normal distance form, i.e.  $A^2 + B^2 = 1$  and  $C$  is the signed distance from the origin to the line. Let  $\mathbf{P} = (x, y)$  be a generic point on a line with unit normal  $\hat{\mathbf{n}} = (A, B)$ , then the equation of the line is  $\hat{\mathbf{n}} \cdot \mathbf{P} = C$ . The orthogonal complement of  $\hat{\mathbf{n}}$ ,  $\hat{\mathbf{n}}^\perp$ , is parallel to the line.

In some constructions, multiple cases and subcases are organized via the following functions.

$$\text{signum}(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \quad \text{signum}_+(x) = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases} \quad \text{signum}_-(x) = \begin{cases} +1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

If  $\mathbf{P}$ ,  $\mathbf{P}_1$  and  $\mathbf{P}_2$  are collinear points with  $\mathbf{P}_1 \neq \mathbf{P}_2$ , then  $\mathbf{P}$  can be expressed as a linear combination of  $\mathbf{P}_1$  and  $\mathbf{P}_2$  using the distances (norms)  $\lambda_1 = \|\mathbf{P}_1 - \mathbf{P}\|$  and  $\lambda_2 = \|\mathbf{P}_2 - \mathbf{P}\|$ . If  $\mathbf{P}$  is between  $\mathbf{P}_1$  and  $\mathbf{P}_2$ , then  $\mathbf{P} = (\lambda_2 \mathbf{P}_1 + \lambda_1 \mathbf{P}_2) / (\lambda_1 + \lambda_2)$ . If  $\mathbf{P}_1$  is between  $\mathbf{P}$  and  $\mathbf{P}_2$ , then  $\mathbf{P} = (\lambda_2 \mathbf{P}_1 - \lambda_1 \mathbf{P}_2) / (-\lambda_1 + \lambda_2)$ . If  $\mathbf{P}_2$  is between  $\mathbf{P}_1$  and  $\mathbf{P}$ , then  $\mathbf{P} = (-\lambda_2 \mathbf{P}_1 + \lambda_1 \mathbf{P}_2) / (\lambda_1 - \lambda_2)$ .

Two constructions use Heron's formula for the area of a triangle to compute the height  $h$  of a triangle. With sides as labelled in Figure 3, the semi-perimeter is  $p_s = (a + b + c) / 2$ . Then the area of the triangle is  $A = \sqrt{p_s(p_s - a)(p_s - b)(p_s - c)} \Rightarrow A = \sqrt{(c^2 - (a + b)^2)((a - b)^2 - c^2)} / 4$ , which gives the height of the triangle as  $h = 2A / c = \sqrt{(c^2 - (a + b)^2)((a - b)^2 - c^2)} / (2c)$ . Again referring to Figure 3, the Law of Cosines is  $a^2 = b^2 + c^2 - 2bc \cos \theta \Rightarrow 2bc \cos \theta = b^2 + c^2 - a^2$ , which implies  $b^2 + c^2 - a^2 > 0 \Leftrightarrow 0 < \theta < \pi/2$  and  $b^2 + c^2 - a^2 < 0 \Leftrightarrow \pi/2 < \theta < \pi$ .

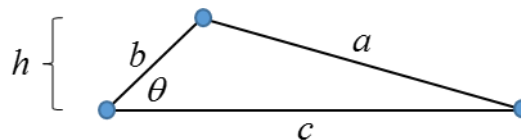


Figure 3: General triangle.

## 3 DERIVATIONS

After introducing some basic ideas (Subsection 3.1), the derivations will be presented in three groups. The first group (Subsection 3.2) contains four derivations relevant to the packing problem.

The second group (Subsection 3.3) contains other tangency constructions relevant to CAD and the final group (Subsection 3.4) contains an additional assortment of constructions relevant to CAD. The constructions are a mixture of the simple and the complex.

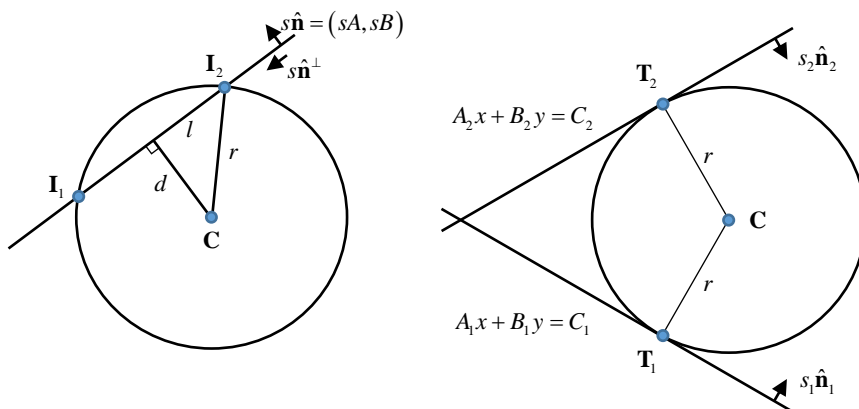
### 3.1 Some Basic Ideas

To introduce the basic ideas of the method, consider a simple problem of finding the intersection of a line and a circle, as shown in Figure 4(a). Let the center of the circle be  $\mathbf{C}$  and let its radius be  $r$ . Let the unit normal to the line be  $\hat{\mathbf{n}}$  and let its signed distance from the origin be  $C$ . Define  $s = \text{signum}_+(\hat{\mathbf{n}} \cdot \mathbf{C} - C)$ . Then  $s\hat{\mathbf{n}}$  points in the direction shown in Figure 4(a), i.e. from the center to the line. Compute  $d = s(\hat{\mathbf{n}} \cdot \mathbf{C} - C)$ , the distance from the center of the circle to the line. There are three cases to consider:  $d > r$ ,  $d = r$ , and  $d < r$ . In the first case, there is no solution. In the second case, there is a single intersection point  $\mathbf{C} + r s\hat{\mathbf{n}}$ , which is also a point of tangency. In the third case, the intersection points are  $\mathbf{I}_1 = \mathbf{C} + d s\hat{\mathbf{n}} + l s\hat{\mathbf{n}}^\perp$  and  $\mathbf{I}_2 = \mathbf{C} + d s\hat{\mathbf{n}} - l s\hat{\mathbf{n}}^\perp$ . For comparison, the traditional approach solves simultaneous equations  $Ax + By = C$  and  $(x - x_c)^2 + (y - y_c)^2 = r^2$ , yielding  $y = (C - Ax)/B$  where

$$x = \left( B^2 x_c + AC - AB y_c \pm \sqrt{(B^2 x_c + AC - AB y_c)^2 - (A^2 + B^2)(B^2(x_c^2 + y_c^2 - r^2) + C^2 - 2BCy_c)} \right) / (A^2 + B^2).$$

Note the following.

- The simple geometric interpretation of the construction of  $\mathbf{I}_1$ : "Start at the center of the circle, go a distance  $d$  in a direction perpendicular to the line, then turn left  $90^\circ$  and go a distance  $l$ ".
- In production code, the test for the second case,  $d = r$ , can be modified easily to use a tolerance, which is awkward to do in the traditional, equation-based solution.
- In an object-oriented language, with appropriate classes defined for points, vectors and circles, and with overloaded arithmetic operators, the formulas yield compact code that is easier to debug since coding errors often can be interpreted visually. Define class *Circle*, with instance *circle*, class *Vector2D*, with instances *n* and *n\_p*, and class *Point2D* with instances *I1* and *I2*. The C# code is  $I1 = \text{circle.Center} + d * s * n + l * s * n\_p$ ; and  $I2 = \text{circle.Center} + d * s * n - l * s * n\_p$ .



**Figure 4:** Simple cases: (a) Intersection of line and circle, and (b) Tangent to two lines.

### 3.2 Derivations Associated with the Packing Problem

#### 3.2.1 Circle tangent to two given lines

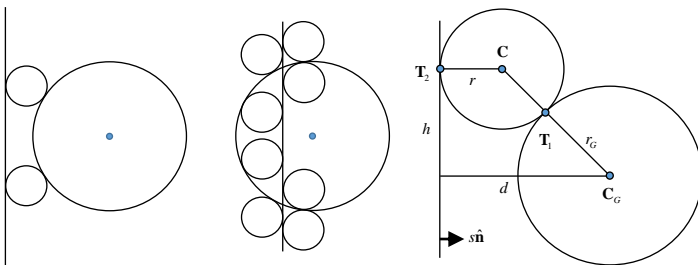
This is the simplest of the circle fittings and continues the introduction of the idea of the method but shows little of its value. Let the given radius be  $r$  and let the normal vectors to the given lines be  $\hat{\mathbf{n}}_1$  and  $\hat{\mathbf{n}}_2$ . Define the values  $s_1 = \text{signum}_+(\hat{\mathbf{n}}_1 \cdot \mathbf{P}_{pick} - C_1)$  and  $s_2 = \text{signum}_+(\hat{\mathbf{n}}_2 \cdot \mathbf{P}_{pick} - C_2)$ . (For the packing application,  $s_1$  and  $s_2$  are specified directly in this, and subsequent, derivations.) The intersecting lines divide the plane into four quadrants and  $s_1\hat{\mathbf{n}}_1$  and  $s_2\hat{\mathbf{n}}_2$  point into the quadrant containing  $\mathbf{P}_{pick}$ , and the constructed circle, as in Figure 4(b).

If line one is translated a distance  $r$  in the direction  $s_1\hat{\mathbf{n}}_1$  and line two is translated a distance  $r$  in the direction  $s_2\hat{\mathbf{n}}_2$ , the center  $\mathbf{C}$  is at the intersection of the translated lines, i.e.  $s_1\hat{\mathbf{n}}_1 \cdot \mathbf{C} = s_1C_1 + r$  and  $s_2\hat{\mathbf{n}}_2 \cdot \mathbf{C} = s_2C_2 + r$ . The equations are solved using Gaussian elimination with partial pivoting to give  $\mathbf{C}$ , which is used to compute the points of tangency as  $\mathbf{T}_1 = \mathbf{C} - rs_1\hat{\mathbf{n}}_1$  and  $\mathbf{T}_2 = \mathbf{C} - rs_2\hat{\mathbf{n}}_2$ . The tangent vector to the center path arc is  $\mathbf{T} = s_1\hat{\mathbf{n}}_1 + s_2\hat{\mathbf{n}}_2$ .

#### 3.2.2 Circle tangent to a given line and a given circle

Let the given radius be  $r$ . Let the given line be  $Ax + By = C$  with unit normal  $\hat{\mathbf{n}}$ . Let the given circle have center  $\mathbf{C}_G$  and radius  $r_G$ . With  $s = \text{signum}_+(\hat{\mathbf{n}} \cdot \mathbf{C}_G - C)$ , the vector  $s\hat{\mathbf{n}}$  points into the half space containing  $\mathbf{C}_G$ . The distance from the given line to the center of the given circle is  $d = s(\hat{\mathbf{n}} \cdot \mathbf{C}_G - C)$ . In the first case, Figure 5(a),  $r_G \leq d$ , the given circle does not intersect the given line (but could be tangent to it). In the second case, Figure 5(b),  $r_G > d$ , the given circle does intersect the given line.

Distinguish the upper and lower subcases using  $s_{ul} = \text{signum}_+((\mathbf{P}_{pick} - \mathbf{C}_G) \cdot s\hat{\mathbf{n}}^\perp)$ . Additional subcases for Figure 5(b) will be identified using  $s_1 = \text{signum}_+(\|\mathbf{P}_{pick} - \mathbf{C}_G\| - r_G)$ ,  $\mathbf{P}_{pick}$  inside or outside of the given circle, and  $s_2 = \text{signum}_+(s(\hat{\mathbf{n}} \cdot \mathbf{P}_{pick} - C))$ ,  $\mathbf{P}_{pick}$  left or right of the given line.



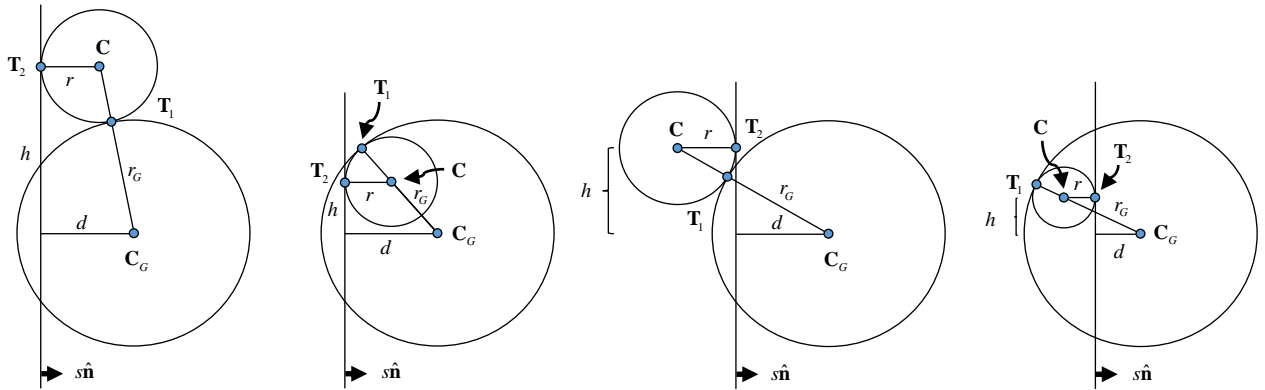
**Figure 5:** Cases: (a) Non-intersecting, and (b) intersecting. **Figure 6:**  $s_1 > 0$  &  $s_2 > 0$ .

Case	Restriction	$s_1$	$s_2$
Figure 6	$2r \geq d - r_G$	+1	+1
Figure 7(a)	none	+1	+1
Figure 7(b)	$2r \leq d + r_G$	-1	+1
Figure 7(c)	none	+1	-1
Figure 7(d)	$2r \leq r_G - d$	-1	-1

**Table 1:** Cases and subcases.

The cases and subcases are summarized in Table 1 and illustrated in Figure 6 and Figure 7. Compute  $h$ , the center  $\mathbf{C}$ , the tangent vector to the center path arc  $\mathbf{T} = d\mathbf{C}/dr$ , and the points of tangency  $\mathbf{T}_1$  and  $\mathbf{T}_2$ :  $h = \sqrt{r_G^2 + s_1 2rr_G - d^2 + s_2 2rd}$ ,  $\mathbf{C} = \mathbf{C}_G + s_{ul}h s\hat{\mathbf{n}}^\perp + s_2(r - s_2d)s\hat{\mathbf{n}}$ ,

$h' = (s_1 r_G + s_2 d) / \sqrt{r_G^2 + s_1 2r r_G - d^2 + s_2 2rd}$ ,  $\mathbf{T} = s_{ul} h' \mathbf{s}\hat{\mathbf{n}}^\perp + s_2 \mathbf{s}\hat{\mathbf{n}}$ ,  $\mathbf{T}_1 = (r_G / (r_G + s_1 r)) \mathbf{C} + (s_1 r / (r_G + s_1 r)) \mathbf{C}_G$ , and  $\mathbf{T}_2 = \mathbf{C} - s_2 r \mathbf{s}\hat{\mathbf{n}}$ .

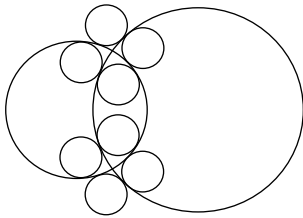


**Figure 7:** Intersecting cases: (a)  $s_1 > 0, s_2 > 0$ , (b)  $s_1 < 0, s_2 > 0$ , (c)  $s_1 > 0, s_2 < 0$ , (d)  $s_1 < 0, s_2 < 0$ .

### 3.2.3 Circle tangent to two given circles

The locations of the constructed circles considered are shown in Figure 8 and Figure 11. ([5] has a complete list of cases and the method works on all of them. For brevity, only those in Figure 8 and Figure 11 are treated here.)

Let  $r$  be the given radius and let the centers of the given circles be  $\mathbf{C}_1$  and  $\mathbf{C}_2$  with radii  $r_1$  and  $r_2$ , respectively. The distance between the centers of the given circles is  $d = \|\mathbf{C}_2 - \mathbf{C}_1\|$  and  $\hat{\mathbf{u}}$  is a unit vector in the direction from  $\mathbf{C}_1$  to  $\mathbf{C}_2$ . The quantities  $s_1, s_2, s_{ul}, s_{dr}$  and  $s_d$ , as defined in Table 2, are used to manage the cases. (For the packing problem,  $s_1, s_2$  and  $s_{ul}$  are specified directly.)



**Figure 8:** Cases Considered.

Inside or outside circle #1	$s_1$	$\text{signum}(\ \mathbf{P}_{pick} - \mathbf{C}_1\  - r_1)$
Inside or outside circle #2	$s_2$	$\text{signum}(\ \mathbf{P}_{pick} - \mathbf{C}_2\  - r_2)$
Above or below center line	$s_{ul}$	$\text{signum}((\mathbf{P}_{pick} - \mathbf{C}_1) \cdot \hat{\mathbf{u}}^\perp)$
Overlapping circles	$s_{dr}$	$\text{signum}_+(d - r_1 - r_2)$
One circle inside other	$s_d$	$\text{signum}_-(d + \min(r_1, r_2) - \max(r_1, r_2))$

**Table 2:** Definitions of case selectors.

First consider the external case  $s_1 > 0$  and  $s_2 > 0$ . The two subcases are  $s_{dr} > 0$  (Figure 9) and  $s_{dr} < 0$ . If  $r_1 < r_2$  then  $\hat{\mathbf{u}}(\mathbf{C} - \mathbf{C}_1)$  can be positive or negative depending on the value of  $r$ . Compute the sign as  $s_b = \text{signum}((r_1 + r)^2 + d^2 - (r_2 + r)^2)$ . (Argument of signum is from the Law of Cosines.) A similar sign change occurs in other cases, but with differing signs in the definition of  $s_b$ .

The other cases where the given circles overlap but one is not contained in the other, are identified as  $s_{dr} < 0$  and  $s_d > 0$ . The cases are shown in Figure 10(a), Figure 10(b) and Figure 10(c). Note also Table 3.

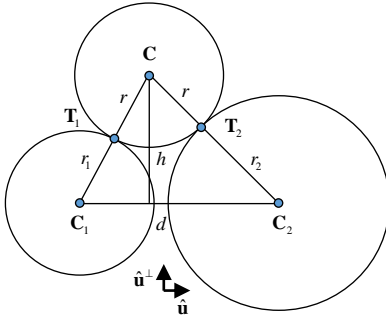


Figure 9: External cases.

Cases	Restrictions	$s_1$	$s_2$	$s_{dr}$	$s_d$
Figure 9	$2r \geq d - r_1 - r_2$	+1	+1	+1	+1
Figure 10(a)	$2r \leq r_1 + r_2 - d$	-1	-1	-1	+1
Figure 10(b)	$2r \leq r_1 - r_2 + d$	-1	+1	-1	+1
Figure 10(c)	$2r \leq -r_1 + r_2 + d$	+1	-1	-1	+1
Figure 11(a)	$r_2 - r_1 - d \leq 2r \leq r_2 - r_1 + d$	+1	-1	-1	-1
Figure 11(b)	$r_1 - r_2 - d \leq 2r \leq r_1 - r_2 + d$	-1	+1	-1	-1

Table 3: Summary of cases.

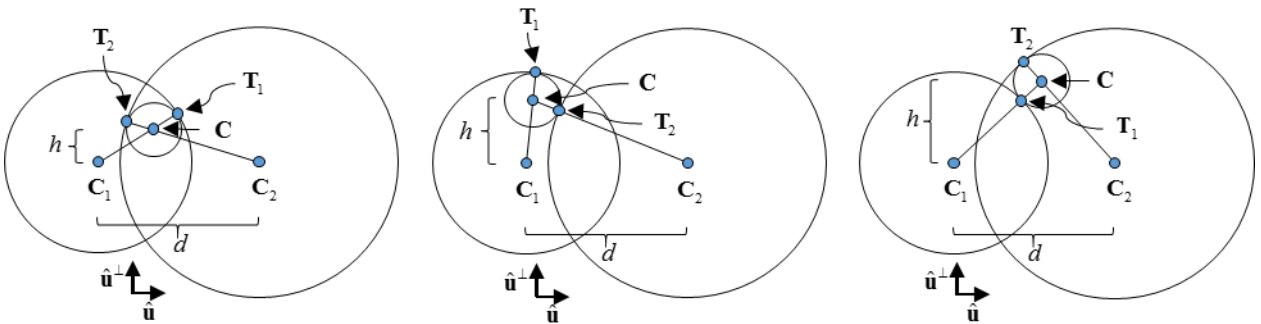


Figure 10: Interior cases: (a) Inside both, (b) Inside 1, outside 2, and (c) Outside 1, inside 2.

A final case is one given circle inside the other, Figure 11. Prior to examining the subcase in Figure 11(b), combine all of the previous cases and subcases, including Figure 11(a).

The calculations are the following. Compute,  $h$ , the height of the triangle formed by  $C$ ,  $C_1$  and  $C_2$ , and compute  $s_b = \text{signum}((r_1 + s_1 r)^2 + d^2 - (r_2 + s_2 r)^2)$ . Use these to compute center of the circle,  $C$ , the tangent vector to the center path  $\mathbf{T} = d\mathbf{C}/dr$ , and the points of tangency,  $\mathbf{T}_1$  and  $\mathbf{T}_2$ , as follows:  $h = \sqrt{(d^2 - (r_1 - s_1 s_2 r_2)^2)((r_1 + s_1 s_2 r_2 + s_1 2r)^2 - d^2)} / (2d)$ ,  $\mathbf{C} = \mathbf{C}_1 + s_b \sqrt{(r + r_1)^2 - h^2} \hat{\mathbf{u}} + s_{ul} h \hat{\mathbf{u}}^\perp$ ,  $h' = s_1 (d^2 - (r_1 - s_1 s_2 r_2)^2) (r_1 + s_1 s_2 r_2 + 2s_1 r) / (2d^2 h)$ ,  $\mathbf{T} = s_b ((r + r_1 - hh') / \sqrt{(r + r_1)^2 - h^2}) \hat{\mathbf{u}} + s_{ul} h' \hat{\mathbf{u}}^\perp$ ,  $\mathbf{T}_1 = (s_1 r / (r_1 + s_1 r)) \mathbf{C}_1 + (r_1 / (r_1 + s_1 r)) \mathbf{C}$ , and  $\mathbf{T}_2 = (s_2 r / (r_2 + s_2 r)) \mathbf{C}_2 + (r_2 / (r_2 + s_2 r)) \mathbf{C}$ .

The formula for Figure 11(b) reverses the roles of 1 and 2 in Figure 11(a). With  $h$  and  $h'$  as above and with  $s_b = \text{signum}((r_2 + r)^2 + d^2 - (r_1 - r)^2)$ , compute the center, tangent vector and tangent points as follows:  $\mathbf{C} = \mathbf{C}_2 - s_b \sqrt{(r_2 + r)^2 - h^2} \hat{\mathbf{u}} + s_{ul} h \hat{\mathbf{u}}^\perp$ ,  $\mathbf{T} = -s_b ((r + r_2 - hh') / \sqrt{(r + r_2)^2 - h^2}) \hat{\mathbf{u}} + s_{ul} h' \hat{\mathbf{u}}^\perp$ ,  $\mathbf{T}_1 = (-r / (r_1 - r)) \mathbf{C}_1 + (r_1 / (r_1 - r)) \mathbf{C}$ , and  $\mathbf{T}_2 = (r / (r_2 + r)) \mathbf{C}_2 + (r_2 / (r_2 + r)) \mathbf{C}$ .

### 3.2.4 Circle tangent to three given lines

Referring to Figure 12, let the given lines be  $\hat{\mathbf{n}}_1 \cdot \mathbf{P} = C_1$ ,  $\hat{\mathbf{n}}_2 \cdot \mathbf{P} = C_2$ , and  $\hat{\mathbf{n}}_3 \cdot \mathbf{P} = C_3$ . Use Gaussian elimination with partial pivoting to compute  $\mathbf{I}_1$ ,  $\mathbf{I}_2$  and  $\mathbf{I}_3$ . Compute  $a = \|\mathbf{I}_2 - \mathbf{I}_1\|$ ,  $b = \|\mathbf{I}_3 - \mathbf{I}_2\|$ , and

$c = \|\mathbf{I}_1 - \mathbf{I}_3\|$ . Then  $d_i$ , the distance from point  $\mathbf{I}_i$  to the nearest points of tangency, is computed as shown in the figure. The points of tangency are computed as  $\mathbf{T}_1 = (d_1/a)\mathbf{I}_2 + (d_2/a)\mathbf{I}_1$ ,  $\mathbf{T}_2 = (d_2/b)\mathbf{I}_3 + (d_3/b)\mathbf{I}_2$ , and  $\mathbf{T}_3 = (d_3/c)\mathbf{I}_1 + (d_1/c)\mathbf{I}_3$ .

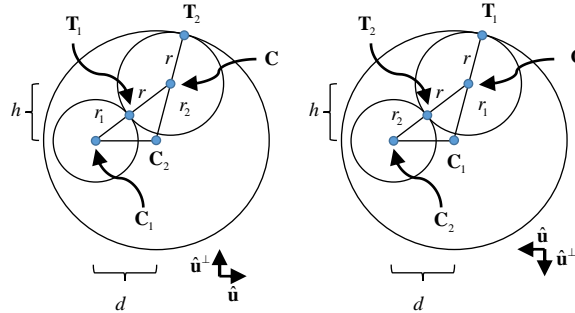


Figure 11: Other enclosing cases: (a) 1 inside 2, (b) 2 inside 1.

The center,  $\mathbf{C}$ , is the intersection of lines through the points of tangency and perpendicular to the tangent lines; solve the following equations in the least squares sense:  $\hat{\mathbf{n}}_1^\perp \cdot \mathbf{C} = \hat{\mathbf{n}}_1^\perp \cdot \mathbf{T}_1$ ,  $\hat{\mathbf{n}}_2^\perp \cdot \mathbf{C} = \hat{\mathbf{n}}_2^\perp \cdot \mathbf{T}_2$ , and  $\hat{\mathbf{n}}_3^\perp \cdot \mathbf{C} = \hat{\mathbf{n}}_3^\perp \cdot \mathbf{T}_3$ . The radius is  $r = \|\mathbf{T}_1 - \mathbf{C}\| = \|\mathbf{T}_2 - \mathbf{C}\| = \|\mathbf{T}_3 - \mathbf{C}\|$ .

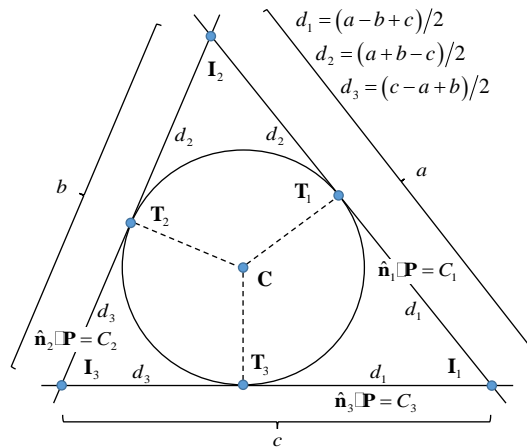


Figure 12: Circle tangent to three lines.

### 3.3 Other Derivations Involving Tangency

#### 3.3.1 Circle through given point and tangent to given line

Let  $r$  be the given radius and let  $\mathbf{P}$  be the given point. Let  $\hat{\mathbf{n}}$  be the unit normal to the given line. Compute  $s = \text{signum}(\hat{\mathbf{n}} \cdot \mathbf{P} - r)$  then compute the quantities in Figure 13. There is a solution if and only if  $d \leq 2r$ . If  $d = 0 \Leftrightarrow s = 0$ , then  $\mathbf{P}$  is on the line, i.e.  $\mathbf{P} = \mathbf{T}$ . Also, note that the center lies on a parabola whose focus is the given point and whose directrix is the given line.

If  $d > 0$ , compute  $\mathbf{P}'$  as shown and compute  $\delta = \sqrt{r^2 - (d - r)^2} = \sqrt{2rd - d^2}$ . The center of the circle can be on either side of the line through  $\mathbf{P}$  and  $\mathbf{P}'$  and the two subcases are resolved by



computing  $s_1 = \text{signum}_+((\mathbf{P}_{pick} - \mathbf{P}) \cdot \mathbf{s}\hat{\mathbf{n}}^\perp)$ . If  $d = 0$ , the circle can be on either side of the given line and the two subcases can be resolved by using  $s_2 = \text{signum}_+(s(\mathbf{P}_{pick} \cdot \hat{\mathbf{n}} - C))$ , giving  $\mathbf{C} = \mathbf{T} + s_2 r s \hat{\mathbf{n}}$ .

All cases can be combined by using  $s_3 = |s| + (1 - |s|)s_2$ , i.e.  $\mathbf{T} = \mathbf{P} - d s \hat{\mathbf{n}} + s_1 \delta s \hat{\mathbf{n}}^\perp$  and  $\mathbf{C} = \mathbf{T} + s_3 r s \hat{\mathbf{n}}$ .

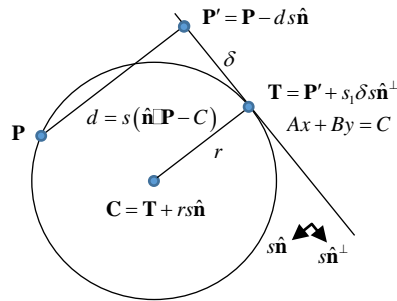


Figure 13: Through point, tangent to line.

### 3.3.2 Circle through given point and tangent to given circle

Let the given radius be  $r$ , let the given point be  $\mathbf{P}$ , let the given circle have center  $\mathbf{C}_G$  and radius  $r_G$ , and let the center of the constructed circle be  $\mathbf{C}$ . Compute the unit vector from  $\mathbf{P}$  to  $\mathbf{C}_G$ ,  $\hat{\mathbf{u}}$ , using the distance is  $d = \|\mathbf{C}_G - \mathbf{P}\|$ . The main cases are identified using  $s = \text{signum}(d - r_G)$ .

If  $s = 0$ , the given point is on the given circle (Figure 14(a)). To identify inside and outside, compute  $s_1 = \text{signum}_+(\|\mathbf{P}_{pick} - \mathbf{C}_G\| - r_G)$ . Note that if  $s_1 < 0$ , there is a solution if and only if  $r < r_G$ . In either case, the center of the circle is computed as  $\mathbf{C} = (-s_1 r / r_G) \mathbf{C}_G + ((r_G + s_1 r) / r_G) \mathbf{P}$ , and the tangency point is the given point.

If  $s \neq 0$ , the given point is not on the given circle (Figure 14(b)). Upper and lower solutions are identified using  $s_2 = \text{signum}_+(\hat{\mathbf{u}}^\perp \cdot (\mathbf{P}_{pick} - \mathbf{C}_G))$ . Consider the triangle formed by the points  $\mathbf{C}$ ,  $\mathbf{P}$  and  $\mathbf{C}_G$ . By Heron's formula, the height is  $h = \sqrt{(d^2 - r_G^2)((2r + s r_G)^2 - d^2)} / (2d)$ . Compute  $s_b = \text{signum}(r^2 + d^2 - (r_G + s r)^2)$ , then the center is  $\mathbf{C} = \mathbf{P} + s_b \sqrt{r^2 - h^2} \hat{\mathbf{u}} + s_2 h \hat{\mathbf{u}}^\perp$  and the point of tangency is  $\mathbf{T} = (s r / (r_G + s r)) \mathbf{C}_G + (r_G / (r_G + s r)) \mathbf{C}$ .

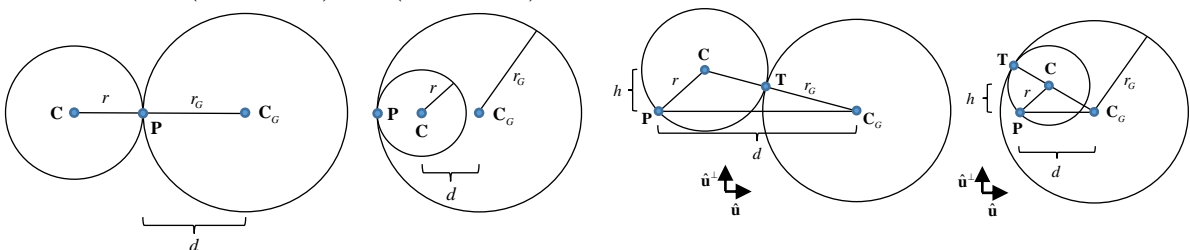


Figure 14: Through point: (a) Given point on given circle, and (b) Given point not on given circle.

### 3.3.3 Circle passing through two given points and tangent to given line

Let  $\mathbf{P}_1$  and  $\mathbf{P}_2$  be the two given points and let  $\hat{\mathbf{n}}$  be the unit normal to the given line, as in Figure 15(a). No assumption is made about which point is closer to the line. Let  $s_1 = \text{signum}(\hat{\mathbf{n}} \cdot \mathbf{P}_1 - C)$  and  $s_2 = \text{signum}(\hat{\mathbf{n}} \cdot \mathbf{P}_2 - C)$ . There is no solution if  $s_1 = s_2 = 0$  (both points are on the line) or if  $s_1 s_2 = -1$  (points are on opposite sides of the line). Compute  $s_{12} = \text{signum}(s_1 + s_2)$ , then  $s_{12} \hat{\mathbf{n}}$ .

There are two cases for the center, based on left-right orientation, which are denoted  $\mathbf{C}_L$  and  $\mathbf{C}_R$  (generically as  $\mathbf{C}$ ). Since the constructed circle is tangent to the line, its radius is  $r = s_{12}(\hat{\mathbf{n}} \cdot \mathbf{C} - C)$ . The center of the circle in either case satisfies the equation of the circle  $\|\mathbf{C} - \mathbf{P}_1\|^2 = \|\mathbf{C} - \mathbf{P}_2\|^2 = (\hat{\mathbf{n}} \cdot \mathbf{C} - C)^2$ .

Definitions for  $\bar{\mathbf{P}}$ ,  $d_p$  and  $l$  are shown in Figure 15(a). Define a unit vector  $\hat{\mathbf{v}}$  from  $\mathbf{P}_1$  to  $\mathbf{P}_2$ . The line segment joining the centers is a perpendicular bisector of the line segment joining  $\mathbf{P}_1$  and  $\mathbf{P}_2$ . Compute  $s = \text{signum}_+((\mathbf{P}_{pick} - \bar{\mathbf{P}}) \cdot \hat{\mathbf{v}}^\perp)$ .

The centers of the circles are expressed as  $\mathbf{C} = \bar{\mathbf{P}} + d \hat{\mathbf{v}}^\perp$  for both cases. If  $s > 0$ ,  $\mathbf{C} = \mathbf{C}_L$  and  $d = d_L$ . If  $s < 0$ ,  $\mathbf{C} = \mathbf{C}_R$  and  $d = -d_R$ . Substituting  $\mathbf{C} = \bar{\mathbf{P}} + d \hat{\mathbf{v}}^\perp$  into  $\|\mathbf{C} - \mathbf{P}_1\|^2 = (\hat{\mathbf{n}} \cdot \mathbf{C} - C)^2$  yields a quadratic equation in the variable  $d$ ,  $ad^2 + bd + c = 0$ , with  $a = 1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}^\perp)^2$ ,  $b = -2(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}^\perp)(\hat{\mathbf{n}} \cdot \bar{\mathbf{P}} - C)$ , and  $c = \|\mathbf{P}_2 - \mathbf{P}_1\|^2 / 4 - (\hat{\mathbf{n}} \cdot \bar{\mathbf{P}} - C)^2$ , where  $a \geq 0$ ,  $c \leq 0$  and  $b$  can be positive, negative or zero. (Recall the usual precautions with the quadratic formula.) The quadratic will have either one or two solutions.

A single solution appears in two situations and yields one circle, so  $\mathbf{P}_{pick}$  is not needed. (i) The coefficient of  $d^2$  is zero, i.e.  $\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}^\perp = \pm 1$ . This means that the line passing through  $\mathbf{P}_1$  and  $\mathbf{P}_2$  is parallel to the given line. In this case, the solution is  $d = -c/b$ . (ii) The discriminant of the quadratic equation,  $b^2 - 4ac$ , is zero. This occurs if one of the points,  $\mathbf{P}_1$  or  $\mathbf{P}_2$ , lies on the given line, and that point is the tangent point. Then  $d = -b/(2a)$ .

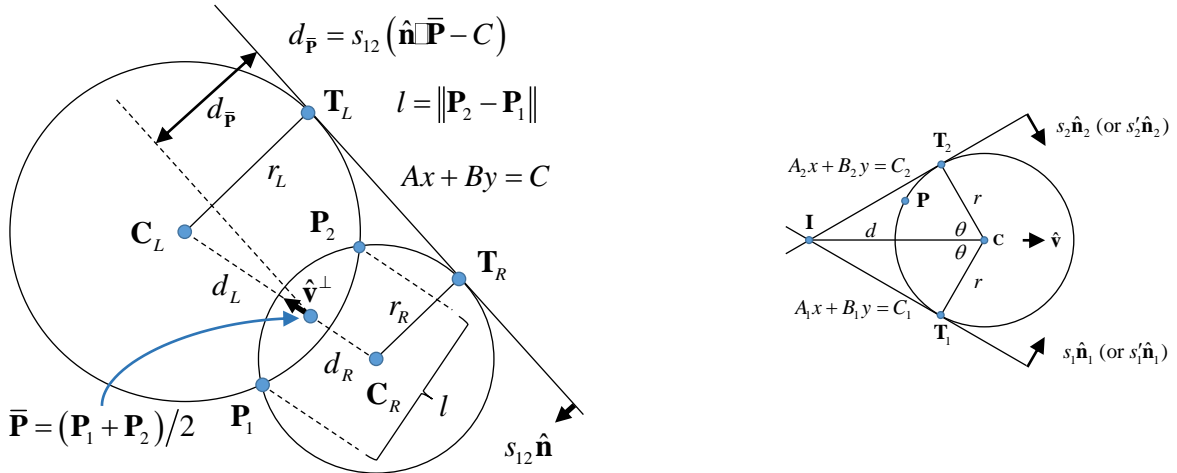
A pair of solutions occurs in two situations. (i) A real conjugate pair occurs if  $b = 0$ , which occurs when  $(\mathbf{P}_2 - \mathbf{P}_1) \cdot s_{12} \hat{\mathbf{n}}$ . Then  $d = \sqrt{-c/a}$ . (ii) Two distinct roots occur, which are given by  $d = (-b \pm \sqrt{b^2 - 4ac}) / (2a)$ . If  $s > 0$  choose  $d > 0$ . If  $s < 0$  choose  $d < 0$ .

So,  $\mathbf{C} = \bar{\mathbf{P}} + d \hat{\mathbf{v}}^\perp$ ,  $r = s_{12}(\hat{\mathbf{n}} \cdot \mathbf{C} - C)$  and the point of tangency is  $\mathbf{T} = \mathbf{C} - r s_{12} \hat{\mathbf{n}}$ .

### 3.3.4 Circle tangent to two given lines and passing through a given point

Let  $\hat{\mathbf{n}}_1$  and  $\hat{\mathbf{n}}_2$  be the unit normals to lines one and two respectively and let  $\mathbf{P}$  be the given point. There are two major cases:  $\mathbf{P}$  is not on either line and  $\mathbf{P}$  is on one of the lines.

Use Gaussian elimination with partial pivoting to compute the intersection point  $\mathbf{I}$  of the two lines then, to manage the cases and subcases, compute:  $s_1 = \text{signum}(\hat{\mathbf{n}}_1 \cdot \mathbf{P} - C_1)$ ,  $s_2 = \text{signum}(\hat{\mathbf{n}}_2 \cdot \mathbf{P} - C_2)$ ,  $s'_1 = \text{signum}_+(\hat{\mathbf{n}}_1 \cdot \mathbf{P}_{pick} - C_1)$ , and  $s'_2 = \text{signum}_+(\hat{\mathbf{n}}_2 \cdot \mathbf{P}_{pick} - C_2)$ . Note that if  $s_1 = 0$  and  $s_2 = 0$ , then  $\mathbf{P} = \mathbf{I}$  and there is no solution.



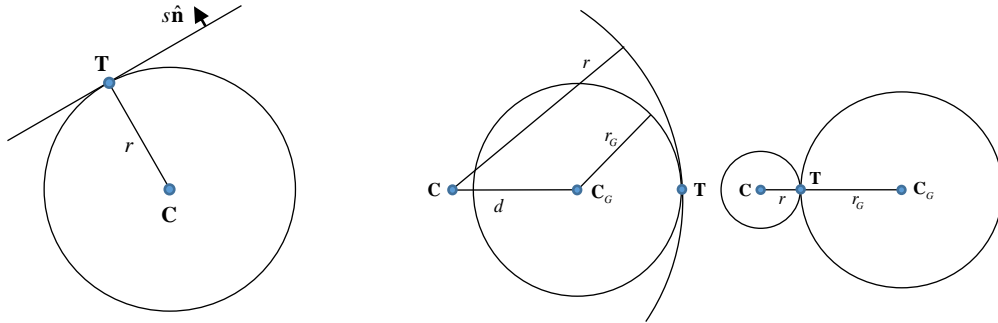
**Figure 15:** Point(s) and tangent line(s): (a) Two points, one line, (b) Two lines, one point.

In the first case, identified as  $s_1 \neq 0$  and  $s_2 \neq 0$ ,  $P$  is not on either line. The intersecting lines divide the plane into four quadrants, one of which contains  $P$ , and  $s_1\hat{n}_1$  and  $s_2\hat{n}_2$  point into this quadrant. The center is  $C = I + d\hat{v}$ , where  $d$  is the distance from  $I$  to  $C$  and  $\hat{v} = (s_1\hat{n}_1 + s_2\hat{n}_2) / \|s_1\hat{n}_1 + s_2\hat{n}_2\|$ . Note that  $r$  and  $d$  are related by  $r = d \cos \theta$ , where  $\cos \theta = \hat{v} \cdot s_1\hat{n}_1 = \hat{v} \cdot s_2\hat{n}_2$ . Since the point  $P$  is on the circle,  $\|P - C\| = r$ . Then  $\|P - C\|^2 = r^2$  can be rewritten as:  $\|P - I\|^2 - 2d\hat{v} \cdot (P - I) + d^2 = d^2 \cos^2 \theta$ . This is a quadratic equation for  $d$ , which in standard form is  $d^2(1 - \cos^2 \theta) + d(-2\hat{v} \cdot (P - I)) + \|P - I\|^2 = 0$ , and can be solved with the quadratic formula. (Recall the standard precautions about using the quadratic formula.) The two solutions correspond to two possible locations for  $C$ . The sign in the quadratic formula is selected using  $s = \text{signum}_+(\hat{v} \cdot (P_{\text{pick}} - P))$ . Since  $\hat{v} \cdot (P - I) > 0$ , the plus sign corresponds to  $s > 0$  and places  $C$  beyond  $P$ , as shown in Figure 15(b). The minus sign corresponds to  $s < 0$ , placing  $C$  between  $I$  and  $P$ . For either choice of sign, the radius is computed as  $r = \|P - C\|$  and the points of tangency are computed as  $T_1 = C - rs_1\hat{n}_1$  and  $T_2 = C - rs_2\hat{n}_2$ .

In the second case,  $P$  is on one of the lines and coincides with one of the tangent points. This case is identified as  $s_1 = 0$  ( $P$  is on line one) or  $s_2 = 0$  ( $P$  is on line two). The constructed circle can be on either side of the line that  $P$  is on. The side is identified using  $P_{\text{pick}}$  (note that  $s'_1\hat{n}_1$  and  $s'_2\hat{n}_2$  point into the quadrant containing  $P_{\text{pick}}$ ). Compute the unit vector  $\hat{v} = (s'_1\hat{n}_1 + s'_2\hat{n}_2) / \|s'_1\hat{n}_1 + s'_2\hat{n}_2\|$ . As in the previous case,  $C = I + d\hat{v}$  and  $r = d \cos \theta$  with  $\cos \theta = s'_1\hat{n}_1 \cdot \hat{v} = s'_2\hat{n}_2 \cdot \hat{v}$ . Noting that point  $P$  is one of the tangent points, Pythagoras' formula yields  $\|P - I\|^2 + d^2 \cos^2 \theta = d^2$ . So,  $d = \|P - I\| / \sqrt{1 - \cos^2 \theta}$  or  $d = \hat{v} \cdot (P - I) / (1 - \cos^2 \theta)$  and  $C = I + d\hat{v}$ ,  $r = \|P - C\|$ , and the points of tangency are  $T_1 = C - rs'_1\hat{n}_1$  and  $T_2 = C - rs'_2\hat{n}_2$ .

### 3.3.5 Circle with given center and given tangent line

Let  $\mathbf{C}$  be the given center point. Let  $\hat{\mathbf{n}} = (A, B)$  be the unit normal to the line  $Ax + By = C$  (Figure 16(a)). Compute  $s = \text{signum}(C - \hat{\mathbf{n}} \cdot \mathbf{C})$ . There is a solution if and only if  $s \neq 0$ . The radius is  $r = s(C - \hat{\mathbf{n}} \cdot \mathbf{C})$  and the point of tangency is  $\mathbf{T} = \mathbf{C} + r s \hat{\mathbf{n}}$ .



**Figure 16:** Center and tangent: (a) Line, and (b) Circle.

### 3.3.6 Circle with given center and tangent circle

Let  $\mathbf{C}$  be the given center point and let  $\mathbf{C}_G$  be the center of the given circle and let  $r_G$  be its radius. Compute the distance between the centers of the circles as  $d = \|\mathbf{C}_G - \mathbf{C}\|$ . Compute a unit vector  $\hat{\mathbf{v}}$  from  $\mathbf{C}$  to  $\mathbf{C}_G$ . Compute  $s_1 = \text{signum}(d - r_G)$  and  $s_2 = \text{signum}(\hat{\mathbf{v}} \cdot (\mathbf{P}_{pick} - \mathbf{C}_G))$ . There are two cases, which are determined by the location of the center of the circle being constructed, i.e. outside ( $s_1 > 0$ , Figure 16(b)) or inside the given circle ( $s_1 < 0$ , not shown), and a special case ( $s_1 = 0$ , also not shown) where the given center point is on the given circle.

For the inside and outside cases, define  $s_3 = \text{signum}(s_1 + s_2 + 1/2)$ . The radius is  $r = s_3(d + s_2 r_G)$  and the point of tangency is  $\mathbf{T} = \mathbf{C} + s_3 r \hat{\mathbf{v}}$ .

In the special case,  $s_1 = 0$ , the radius of the circle is  $r = d + r_G = 2r_G$  with point of tangency  $\mathbf{T} = \mathbf{C} + r \hat{\mathbf{v}}$ .

### 3.3.7 Line tangent to circle through point on circle

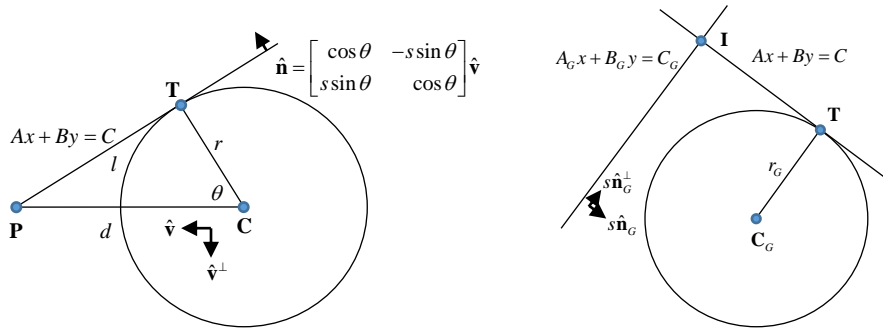
Let the center of the given circle be  $\mathbf{C}$  and let its radius be  $r$  and let  $\mathbf{P}$  be the given point. The unit normal to the line is  $\hat{\mathbf{n}} = (\mathbf{P} - \mathbf{C}) / \|\mathbf{P} - \mathbf{C}\| = (A, B)$  and  $C = \hat{\mathbf{n}} \cdot \mathbf{C} + r$ , which completes the construction.

### 3.3.8 Line through given point and tangent to given circle

Let the given point be  $\mathbf{P}$  and let the center of the given circle be  $\mathbf{C}$  and its radius be  $r$ . Let the distance from the center to the point be  $d = \|\mathbf{P} - \mathbf{C}\|$ . Note that there is a solution if and only if  $d > r$ . (If  $d = r$ , the point is on the circle and is covered by the construction in sub-subsection 3.3.7.) Define a unit vector  $\hat{\mathbf{v}}$  from  $\mathbf{C}$  to  $\mathbf{P}$ .

There are two cases to consider. One case is shown in Figure 17(a). In the other case, the constructed line is reflected in the line from the center to the point. In both cases, the directions of  $\hat{\mathbf{v}}$  and  $\hat{\mathbf{v}}^\perp$  are the same.

Compute  $s = \text{signum}_+((\mathbf{P}_{pick} - \mathbf{C}) \cdot \hat{\mathbf{v}}^\perp)$ . Note that  $\sin \theta = l/d$  and  $\cos \theta = r/d$ . The normal to the line,  $\hat{\mathbf{n}}$ , can be obtained by rotating vector  $\hat{\mathbf{v}}$  through an angle  $-\theta$  for the case shown in Figure 17(a), and through an angle  $+\theta$  for the other case. The construction of the line is completed by computing  $C = \hat{\mathbf{n}} \cdot \mathbf{P}$ . The point of tangency is  $\mathbf{T} = \mathbf{C} + r\hat{\mathbf{n}}$ .



**Figure 17:** Line tangent to circle: (a) Through point, and (b) Perpendicular to line.

### 3.3.9 Line tangent to given circle and perpendicular to given line

Let the given circle have center  $\mathbf{C}_G$  and radius  $r_G$ . Let the given line have unit normal  $\hat{\mathbf{n}}_G$ . Compute  $s = \text{signum}_+(\hat{\mathbf{n}}_G \cdot \mathbf{C}_G - C_G)$ , then  $s\hat{\mathbf{n}}_G$  points in the direction shown in Figure 17(b). To determine the location of the constructed line, compute  $s_p = \text{signum}_+(s\hat{\mathbf{n}}_G^\perp \cdot (\mathbf{P}_{pick} - \mathbf{C}_G))$ . The point of tangency is  $\mathbf{T} = \mathbf{C}_G + s_p r_G s\hat{\mathbf{n}}_G^\perp$ . The normal to the constructed line is  $\hat{\mathbf{n}} = (A, B) = s\hat{\mathbf{n}}_G^\perp$  and the construction is completed by computing  $C = \hat{\mathbf{n}} \cdot \mathbf{T}$ . The intersection point of the lines is  $\mathbf{I} = \mathbf{T} - d s\hat{\mathbf{n}}_G$ , where  $d = s(\hat{\mathbf{n}}_G \cdot \mathbf{C}_G - C_G)$ .

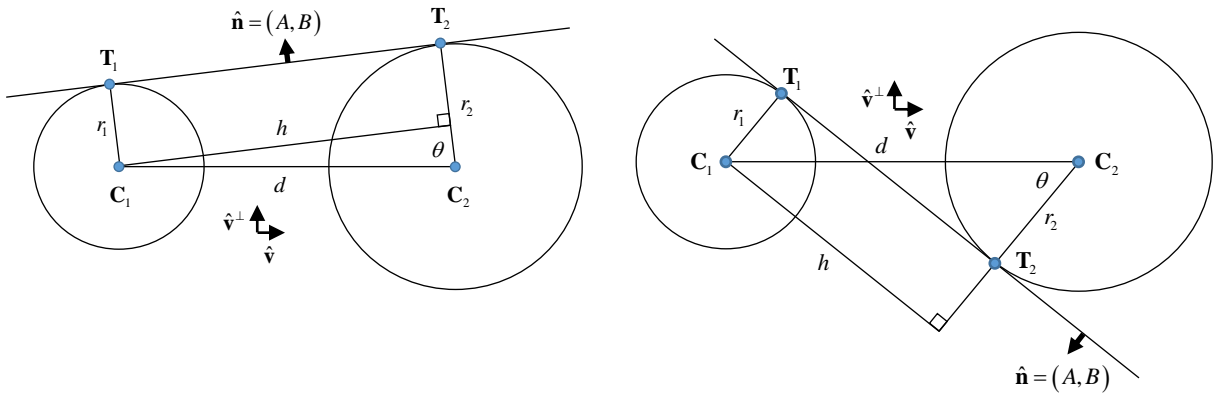
### 3.3.10 Line tangent to two given circles

Let the centers of the circles be  $\mathbf{C}_1$  and  $\mathbf{C}_2$ , and with radii  $r_1$  and  $r_2$ , respectively. Compute the distance between the centers as  $d = \|\mathbf{C}_2 - \mathbf{C}_1\|$  and compute a unit vector  $\hat{\mathbf{v}}$  in the direction from  $\mathbf{C}_1$  to  $\mathbf{C}_2$ . Compute  $s_1 = \text{signum}_+((\mathbf{P}_{pick_1} - \mathbf{C}_1) \cdot \hat{\mathbf{v}}^\perp)$  and  $s_2 = \text{signum}_+((\mathbf{P}_{pick_2} - \mathbf{C}_2) \cdot \hat{\mathbf{v}}^\perp)$ .

The two major cases depend upon whether or not the constructed line crosses the line joining the centers of the circles and subcases depend on the location of the tangent points relative to the line joining the centers, i.e. which side. Figure 18(a) and Figure 18(b) show a subcase of each case. (A special case where the circles are tangent is treated later.) The general case associated with Figure 18(a) has  $s_1 s_2 > 0$  and the general case associated with Figure 18(b) has  $s_1 s_2 < 0$ . The case in Figure 18(a) has a solution if and only if  $d + \min(r_1, r_2) > \max(r_1, r_2)$ . The case in Figure 18(b) has a solution if and only if  $d > r_1 + r_2$ .

The normal,  $\hat{\mathbf{n}}$ , to the constructed line is obtained by rotating the unit vector  $\hat{\mathbf{v}}$  and the rotation matrix can be expressed in terms of the angle  $\theta$ . Note that depending on the relative magnitudes of the radii,  $\theta$  will be measured on either the first or the second circle. Also note that  $\sin \theta$  and  $\cos \theta$  can be expressed in terms of the lengths of the sides of the triangle containing  $\theta$ .

All cases and subcases can be combined as follows. Compute  $h = \sqrt{d^2 - (r_2 - s_1 s_2 r_1)^2}$  then compute  $\hat{\mathbf{n}}$  as  $\hat{\mathbf{n}} = \begin{bmatrix} s_1 s_2 (r_1 - s_1 s_2 r_2)/d & -s_2 h/d \\ s_2 h/d & s_1 s_2 (r_1 - s_1 s_2 r_2)/d \end{bmatrix} \hat{\mathbf{v}}$ . The points of tangency are  $\mathbf{T}_1 = \mathbf{C}_1 + s_1 s_2 r_1 \hat{\mathbf{n}}$  and  $\mathbf{T}_2 = \mathbf{C}_2 + r_2 \hat{\mathbf{n}}$ . The construction of the line is completed by computing  $C = \hat{\mathbf{n}} \cdot \mathbf{T}_1$ .



**Figure 18:** Line tangent to two circles: (a) Same side, and (b) Opposite sides.

A final case occurs when the circles are tangent. One subcase is identified as  $d = r_1 + r_2$ , neither circle is contained in the other, and the other two subcases, one circle is contained inside the other, are identified as  $d = \max(r_1, r_2) - \min(r_1, r_2)$ . All three subcases can be combined as follows. Set  $\hat{\mathbf{n}} = \hat{\mathbf{v}}$ . For the subcase with non-overlapping circles, set  $s = +1$ . For the subcase of one circle inside another, set  $s = -1$ . The point of tangency is computed as  $\mathbf{T} = (r_2 / (s r_1 + r_2)) \mathbf{C}_1 + (s r_1 / (s r_1 + r_2)) \mathbf{C}_2$  and the construction is completed by computing  $C = \hat{\mathbf{n}} \cdot \mathbf{T}$ .

### 3.4 Other Derivations, Mostly Simple

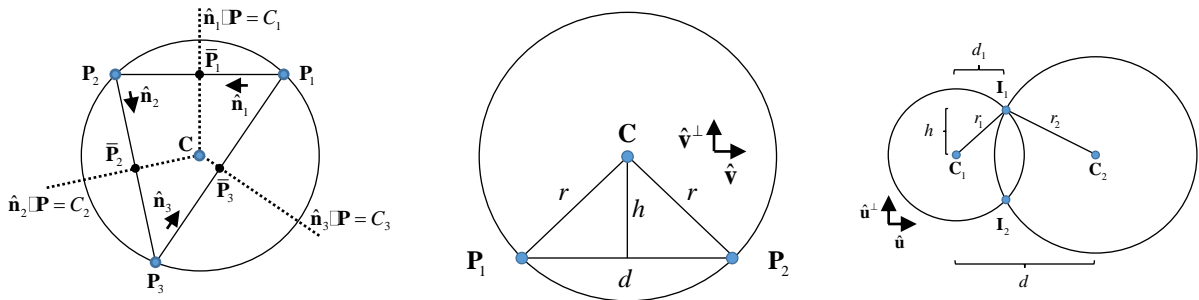
#### 3.4.1 Circle through three points

The center of the circle is located at the intersection point of the perpendicular bisectors of the line segments joining the points. Without loss of generality, assume that the points are in counter-clockwise order.

Define the unit vectors  $\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3$  as shown in Figure 19(a). Compute the midpoints of the sides of the triangle whose vertices are  $\mathbf{P}_1, \mathbf{P}_2$  and  $\mathbf{P}_3$  as  $\bar{\mathbf{P}}_1 = (\mathbf{P}_1 + \mathbf{P}_2)/2$  and similarly for  $\bar{\mathbf{P}}_2$  and  $\bar{\mathbf{P}}_3$ . Compute  $C_1 = \hat{\mathbf{n}}_1 \cdot \bar{\mathbf{P}}_1$  and similarly  $C_2$  and  $C_3$  to complete the definitions of the dotted lines. The equations of the perpendicular bisectors of the line segments are  $\hat{\mathbf{n}}_1 \cdot \mathbf{P} = C_1$ , etc., where  $\mathbf{P}$  is a general point on the line.  $\mathbf{C}$  lies at the common intersection point of the perpendicular bisectors, which is computed in a least squares sense to minimize the error and to avoid the decision of which two equations are the best to solve. The radius of the circle is  $r = \|\mathbf{P}_1 - \mathbf{C}\| = \|\mathbf{P}_2 - \mathbf{C}\| = \|\mathbf{P}_3 - \mathbf{C}\|$ .

### 3.4.2 Circle with given radius and passing through two given points

Compute the distance between the points,  $d = \|\mathbf{P}_2 - \mathbf{P}_1\|$ . There is a solution if and only if  $0 < d \leq 2r$  (Figure 19(b)). Compute a unit vector  $\hat{\mathbf{v}}$  from  $\mathbf{P}_1$  to  $\mathbf{P}_2$ ,  $s = \text{signum}_+((\mathbf{P}_{pick} - \mathbf{P}_1) \cdot \hat{\mathbf{v}}^\perp)$  and  $h = \sqrt{r^2 - (d/2)^2}$ . The center of the circle is  $\mathbf{C} = \mathbf{P}_1 + (d/2)\hat{\mathbf{v}} + sh\hat{\mathbf{v}}^\perp$ .



**Figure 19:** Point constructions: (a) Through three points, (b) Through two points, and (c) Intersection.

### 3.4.3 Intersection of two circles

Let the circles have centers  $\mathbf{C}_1$  and  $\mathbf{C}_2$  and radii  $r_1$  and  $r_2$ , respectively. Compute the distance between the centers as  $d = \|\mathbf{C}_2 - \mathbf{C}_1\|$ . If  $|r_2 - r_1| < d < r_1 + r_2$ , there are two intersection points, as shown in Figure 19(c). Define a unit vector  $\hat{\mathbf{u}}$  pointing from  $\mathbf{C}_1$  to  $\mathbf{C}_2$ , then compute  $d_1 = (r_1^2 - r_2^2 + d^2)/(2d)$  and  $h = \sqrt{r_1^2 - d_1^2}$ . The intersection points are  $\mathbf{I}_1 = \mathbf{C}_1 + d_1\hat{\mathbf{u}} + h\hat{\mathbf{u}}^\perp$  and  $\mathbf{I}_2 = \mathbf{C}_1 + d_1\hat{\mathbf{u}} - h\hat{\mathbf{u}}^\perp$ .

### 3.4.4 Line between two points and equidistant from them

Let  $\mathbf{P}_1$  and  $\mathbf{P}_2$  be the given points. Compute the line's normal  $\hat{\mathbf{n}} = (\mathbf{P}_2 - \mathbf{P}_1) / \|\mathbf{P}_2 - \mathbf{P}_1\| = (A, B)$  then complete the construction of the line with  $C = \hat{\mathbf{n}} \cdot \mathbf{P}$  where  $\mathbf{P} = (\mathbf{P}_1 + \mathbf{P}_2)/2$ .

### 3.4.5 Line through a point and perpendicular to a given line

Let  $A_Gx + B_Gy = C_G$  be the given line with normal  $\hat{\mathbf{n}}_G = (A_G, B_G)$  and let  $\mathbf{P}_G$  be the given point. The normal to the constructed line is  $\hat{\mathbf{n}} = (A, B) = \hat{\mathbf{n}}_G^\perp = (-B_G, A_G)$  and  $C = \hat{\mathbf{n}} \cdot \mathbf{P}_G$ , which completes the construction.

### 3.4.6 Line parallel to a given line at a given distance

Let  $A_Gx + B_Gy = C_G$  be the given line with unit normal  $\hat{\mathbf{n}}_G = (A_G, B_G)$  and let  $d_G$  be the given distance. The lines are parallel, so their normals are parallel, the normal to the constructed line is  $\hat{\mathbf{n}} = (A, B) = \hat{\mathbf{n}}_G$ . With  $s = \text{signum}_+(\hat{\mathbf{n}}_G \cdot \mathbf{P}_{pick} - C_G)$ , complete the construction with  $C = C_G + sd_G$ .

## 4 SUMMARY

The method presented herein, offers a simple, compact and visually intuitive way to compute circle tangencies needed in circle packing and offers an alternative to traditional CAD constructions, yielding formulas that can be manipulated analytically. It is important to note that these are new constructions and not just a re-packaging of old constructions in terms of vectors. In all cases, the methods are simpler to code and test in a modern, object-oriented language than are traditional methods. Specifically, the methods yield very compact code that is easier to understand and easier to debug since the source of a coding error often can be inferred from the error in the displayed result. Finally, with proper classes for points, vectors, lines and circles, the details of the underlying coordinate system are encapsulated in the constructors of the classes and, as suggested by the derivations, the formulas are independent of the coordinate system, i.e. no assumptions of the form “assume the center of the circle is ...” or “assume the orientation is ...”

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## REFERENCES

- [1] Castillo, I.; Kampas, F. J.; Pintér, J. D.: Solving circle packing problems by global optimization: Numerical results and industrial applications, *European Journal of Operational Research*, 191 (3), 2008, 786-802. <https://doi.org/10.1016/j.ejor.2007.01.054>
- [2] Hifi, M.; M'Hallah, R.: A Literature Review on Circle and Sphere Packing Problems: Models and Methodologies, *Advances in Operations Research*, Volume 2009, Article ID 150624, <http://dx.doi.org/10.1155/2009/150624>
- [3] Sugihara, K.; Sawai, M.; Sano, H.; Kim, D.S.; Kim, D.: Disk packing for the estimation of the size of a wire bundle, *Japan Journal of Industrial and Applied Mathematics*, 21(3), 2004, 259-278, <https://doi.org/10.1007/BF03167582>
- [4] Bowyer, A.; Woodwark, J.: *A Programmer's Geometry*, Butterworths, 1983.
- [5] Rogers, D. F.: Interactive graphics and numerical control, *Computer-Aided Design*, 12(5), 1980, 253-261. [https://doi.org/10.1016/0010-4485\(80\)90030-5](https://doi.org/10.1016/0010-4485(80)90030-5)