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# A hybrid approach to define and represent material distribution in heterogeneous objects

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**ABSTRACT**
A hybrid representation is proposed to overcome the current limitations in modeling heterogeneous objects. The representation is based on the idea of decomposing the geometry such that the existing class of material functions (distance based function) can be used to define desired material variation in heterogeneous objects. This representation is supported by Medial Axis Transform (MAT), which defines and represents the material variation from inside to outside for heterogeneous objects. Local symmetry inherent in the MAT and the relationship between the object boundary and the MAT point has been used to achieve intuitive variations of material composition, local control over the same and a compact data structure. A material evaluation scheme has been proposed to evaluate the material composition at any given location in the object. Heterogeneous objects modeled with this scheme are shown in results depicting the material distribution varying from inside to outside and vice versa. This representation is general to any object because of the existence of a unique MAT for all objects.

**KEYWORDS**
Heterogeneous Object Model; Medial Axis Transform; interpolation

## 1. Introduction

In the last two decades, push from the advances in manufacturing such as Layered manufacturing on the one hand and the pull from the need for complex, high performance materials in applications such as biomedical devices, and aerospace have resulted in objects that exhibit multiple functionalities like high hardness and high toughness simultaneously with optimum weight. The multiple and often conflicting material behaviors have been achieved by the use of multiple materials within an object. Such objects are referred to as heterogeneous objects. The variation of material in such objects can be discrete or continuous as shown in Figure 1. Figure 1(a) shows a simplified example of cutting tool with two distinct materials regions shown in blue and red separated by a visible boundary. Figure 1 (b) shows the continuous and gradual changes between two material regions. The latter type of variation in the material composition (hence, the material properties) results in what are called functionally graded materials. Construction of computer models and representation of such objects is the focus of the paper.

Heterogeneous objects have been modeled using evaluated model or unevaluated models [11]. Evaluated models decompose the geometry of the object into simple cells and define the material composition within each cell; unevaluated model maps the material function to the geometry of the object. Evaluated models have the potential to represent wide range of heterogeneous objects by subdividing the geometry and define the material distribution in the sub-divided domains. However, the sequential approach of decomposition and defining the material distribution leads to inaccuracy in material distribution and geometry for the coarse mesh size and large memory size for finer mesh size. Moreover, editing the material distribution requires decomposition of the original geometry from scratch which is expensive. On the other hand, unevaluated models use analytic functions to define geometry and material composition that can be stored compactly and edited efficiently. But unevaluated models can only model a limited range of heterogeneous objects because of the limited functions. These functions are defined in terms of the shape parameters or distance from the material reference features. Distance from material reference feature defines material distribution independent of the shape which models the material properties more realistically. The problem of heterogeneous material modeling is seen as a boundary value problem where the user prescribes a known material composition at specified entities (either on or inside the object) and the modeling scheme has to then blend the different compositions to realize a smooth variation

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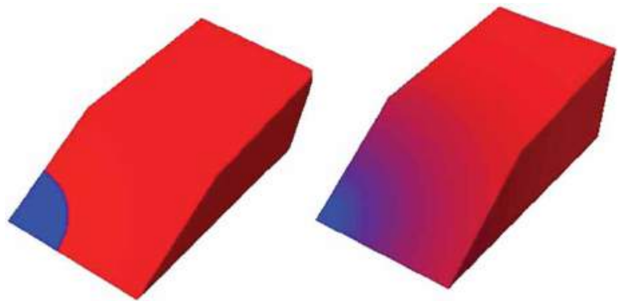

**Figure 1.** Cutting tool having (a) Discrete material distribution (b) Continuous material distribution.

elsewhere within the object [3]. When the interest of the user is in having a simple variation in the composition from the boundary into the object or the other way, this approach requires the user to either discretize the domain to obtain entities inside the domain or to introduce new material features inside the domain to define the material composition. This paper develops a hybrid representation that is based on the Medial Axis Transform (MAT). The relationship between MAT points and corresponding boundary points is used to define the desired material gradient from inside to outside, adaptively discretize the domain in the direction of material blending, and build a smooth representation. Thus, it can model the inside out material distribution intuitively, subdivide the domain according to the material distribution, which overcomes limitation of evaluated model and can be subdivided to any level of resolution like an unevaluated model.

## 2. A brief review of heterogeneous object modeling schemes

Heterogeneous Object Model (HOM) defines a material function over a geometric domain and represents them in either evaluated model or unevaluated model as described in Kou and Tan [11].

*Evaluated model:* Evaluated model subdivides the geometric domain exhaustively into regular sets and defines material function over each regular set. Material distribution over these sets is achieved by material functions that are either a distance function from the material reference features (principal axes, planes; cylindrical axis, sphere center etc) or analytic functions resulting from optimization and simulation.

Evaluated models are required to use numerical simulation (like Finite Element Analysis) to design material and optimize the shape of the object. Jackson et al. [9] divided the complex solid using tetrahedron decomposition methods, and in each decomposed sub-region, the material composition is obtained using analytic blending functions. Cho and Ha [6] used quadratic elements (i.e. pixels) to discretize the geometric domain and used an optimization scheme to determine the material composition of each element for relaxing thermal stress. Gupta and Tandon [8] used the primitives based on convolution to construct the complex variations in heterogeneous object model. In all these methods, spatial decomposition using mesh element or voxel is done prior to defining the material function. So, the approximation of the material function may lead to generation of large number of mesh elements and expensive material interrogation.

*Unevaluated model:* Unevaluated model defines the material function directly over the geometric domain without prior decomposition. Unevaluated models are compact, analytic and efficient to interrogate the material distribution. Geometric domain has functional representation (B-rep, f-rep). The material functions may be parametric function of form feature, distance based functions defined using reference features or any other analytical functions.

Samanta and Koc [20] used the surface parameters to define the material functions for free form surfaces. Biswas et al. [3] proposed an intuitive means of modeling desired material distributions by using distance function from material reference features. The authors utilize the theory of R-functions [19] to construct smooth approximations to distance function for semi-analytic features. Based on the approximation, a distance based function with a canonical form is used to formulate the material distributions. Qian and Dutta [17] used the B-spline representation for the heterogeneous object and developed physics based method (diffusion process) to control and define various material composition profiles. All these methods are more useful to control the design of geometry rather than material composition as the material representation is tightly coupled with geometric representation. This limits the user's degree of freedom in prescribing material composition. Feature Tree [HFT] structure to represent different types of material gradations. HFT structure stored geometric and material transitions from one-dimensional entity to two-dimensional entity and from two-dimensional entity to three-dimensional entity. This method can tailor complex material distribution for simple geometries like sweep, extrusion etc.

Ozbolat and Koc [14] presented a feature based method to represent and design heterogeneous objects with material composition varying along multiple directions. They constructed the Voronoi diagram and the variation of the material composition from the bounding curve to internal curves is obtained using an optimization approach. The optimization approach uses visibility constraints to match point on one bounding curve to another internal curve. These matching lines (ruling lines) do not represent minimum distance from bounding curve. So, the distance based material function cannot be mapped along the ruling lines. In addition, Voronoi diagram is not suitable to map across multiple features because large number of voronoi segments will get generated depending on the sampling of points on bounding curve and some Voronoi segments may lie far from surface of the object in three-dimension [7].

Pasko et al. [15] represented heterogeneous object as multidimensional point sets with multiple attributes based on a function representation (FRep). FRep is used as the basic model for point set geometry. The attributes, i.e. material composition, is represented independently using real-valued scalar function. Yoo [26] proposed the use of radial basis function to model heterogeneous objects.

Both evaluated and unevaluated representation schemes may not provide the material features to meet the given design objectives and constraints. Various numerical based techniques [6, 10, 17] have been proposed to search for the best solution for the given design parameters. Wang and Wang [25] proposed a level set based optimization technique to define the continuous material property distribution, which incorporate various geometric, topological, and material properties within a variation approach.

*Limitations of the current art can be summarized as follows:*

i. The sequential approach of evaluated model [9, 6], i.e. exhaustive decomposition of the geometry into simpler cells and definition of material composition function over them, leads to discretization error in geometry and approximation of the material distribution in the subdivided elements. Any change in smoothness of the material function requires regeneration of grids and re-approximation of the material distribution.
ii. Unevaluated models [3, 20, 12, 15, 23] suffer from the ability to handle only a limited set of functions to represent the desired material distribution.
iii. Most of the unevaluated models [20, 17, 12] use the geometric parameters or geometric form features (using cylindrical coordinates, spherical coordinates, tri-variate splines) to model material distribution. This results in constraining the user to control the material distribution in terms of the geometric entities available to define the geometric form. Distance functions can more realistically model the desired material property distribution. However, the computation of Euclidean distance is slow and inaccurate. More importantly constructing an evaluated representation (required for finite elements for numerical simulation) is not very straightforward as the representation lacks topological information for adaptive discretization of the geometry.

## 3. Background

### 3.1. Material feature

Analogous to shape features [2], a material feature is defined as representation of the material composition mappable to the given object. In this paper, material composition is defined as a function of distance from a material reference entity. Thus, the material feature consists of two quantities: material reference entity and the material distribution function. Material features are useful to generate variant material distribution.

### 3.2. Material composition vector and material composition function

The terms material composition vector and material composition function as used in this paper are based on the definitions of Siu and Tan [24]. Material composition vector represents the volume fraction of each constituent material at any geometric point where the volume fraction of the constituent materials forms the basis of the material space.

At any point (x, y, z), material composition vector is denoted by $M = [m_1,\ m_2,\ m_3,\ \ldots .m_n]$ subject to the constraint of $m_1 + m_2 + m_3 + \ldots m_n = 1$ where n is the number of constituent materials and $m_i$ is the volume of fraction of the i$^{th}$ constituent material. Each component in the material composition vector is defined as a function of distance from some reference entity. These functions are referred to as the material composition functions [24].

The terms material composition vector and material coordinate, and the terms material composition function and material distribution function are used interchangeably in this paper.

### 3.3. Representation

In this paper, Heterogeneous Object Model (O) is defined as the tuple of the material reference entities (G), material composition function (A), discretization factor (N) for approximating material composition function.

$$O = (G, A, N) \tag{1}$$

This is evaluated form for the representation due to discretization of the geometry and material function. If the representation stores only exact form of geometry and material functions i.e. the material reference entities (G) and material composition function (A) without discretization factor (N), then this representation is called unevaluated. Thus, this representation is named hybrid of both evaluated and unevaluated model to represent HOM.

Operating over this representation O, physical quantities like density, temperature, stress, strain etc. can be evaluated for multi-material or functionally graded objects. In this paper, the representation constructed for HOM will be used for efficient querying only.

### 3.4. Decomposition of domain using MAT

MAT is the locus of the center of the maximal disc/ball, which touches the boundary at the foot-points (point of tangency). Figure 2 shows foot-points a’, b’ and c’. Blum and Nagel [4] subdivided the MAT based on different type of points that are normal point, branch point and end point.

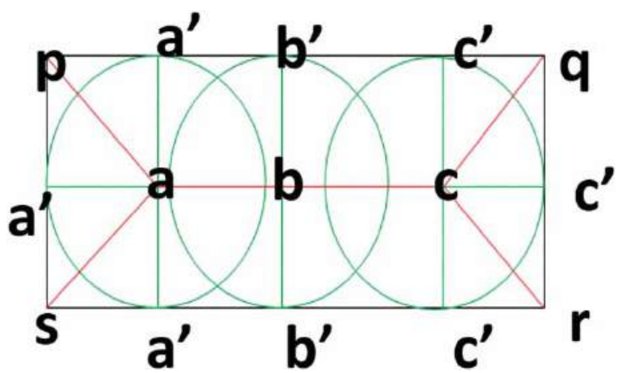


**Figure 2.** 2D medial axis transform.

i. **A normal point:** A point whose maximal disc touches the object border in exactly two separate contiguous sets of points.
ii. **A branch point:** A point whose maximal disc touches the object border in three or more separate contiguous sets; Figure 2 has 2 branch points (a, c).
iii. **An end point:** A point whose maximal disc touches the object border in exactly one contiguous set; Figure 2 has 4 end points (p, q, r, s).

Subdivided MAT bounded by these points is referred as MAT segments. Figure 2 shows the red lines indicating the medial axis, the black lines the boundary of the object and the green circles maximal circles, with their centers at a, b, c, and their corresponding foot points at a’, b’ and c’ respectively.

The line joining the foot-point to the MAT point is called rail. The region bounded by the pair of adjacent rails is called tracks (Refer to Figure 3 (a)). Quadrilateral elements can be obtained by placing sleepers/ties at the appropriate spacing along the rails from the boundary towards the interior till the medial axis as shown in Figure 3(b). The Laytracks algorithm [18] is used for generating these elements.

## 4. Overview of approach

The input to the construction and representation of the heterogeneous object are the following: representation of the shape of the object (either a Boundary Representation or a piece wise linear complex) and the MAT of the object. The representation of the MAT that is input is a collection of points on the medial axis along with its corresponding foot points on the boundary of the domain. The sampling of the MAT points is assumed to be fine enough that

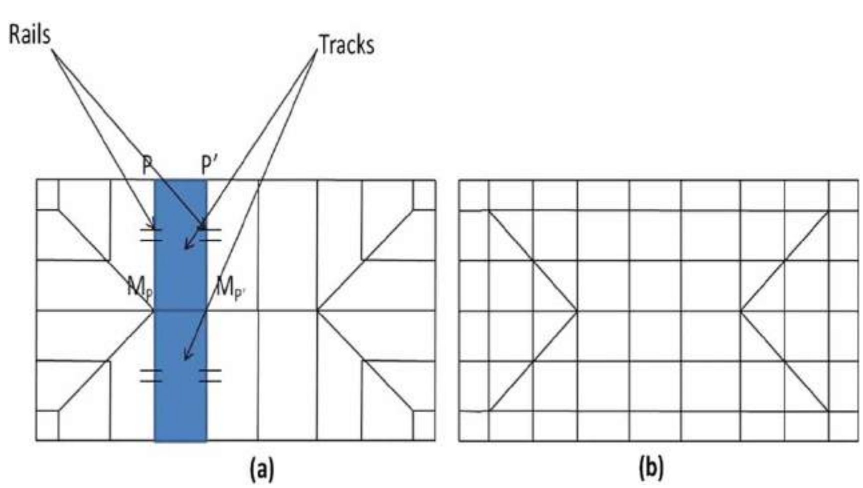


**Figure 3.** Segmentation of a 2D domain using rails.

the rails can be obtained at any desired level of resolution. The user then specifies material composition at the boundary and the MAT. By default, the material variation between the MAT and boundary is linearly interpolated along the rail as shown in Figure 3 (a). Linear interpolation is efficient and widely supported for rendering (rendering is achieved by mapping material composition components to appropriate colors). However, the user can also define the material variation by choosing higher order of interpolation schemes (like quadratic, cubic). The geometric coordinates and material coordinates (or material composition vector) are stored for the start and end point of the rail along with the material variation function along the rail. This forms the unevaluated form of the proposed representation described in section 5. As MAT enables a structured decomposition of the domain through rails into tracks that can be subdivided into mesh elements. The resulting elements with material composition vector at the nodes can be stored and used for analysis, rendering and manufacturing. This forms the evaluated form of the proposed representation described in section 6. Thus, the proposed method is hybrid of both evaluated and unevaluated representations.

As shown in Figure 4(a) and (b), the first step is to specify the material composition at the material reference entities that are boundary and MAT. The material composition on the boundary and MAT are M1 and M2 respectively. The second step is to define the material composition function (shown in Figure 5) over the rails. Figure 4(c) shows the material composition captured along the rails as prescribed (varying from blue at one end to green at the other depicting the two material compositions at the boundary and MAT respectively) after defining material composition function. Finite numbers of rails are shown for the sake of simplicity. Figure 4(d) shows a rendered view of the material composition in the object. While the composition at any point is obtained exactly, the rendering is obtained by interpolating the composition over a track bounded by the adjacent rails.

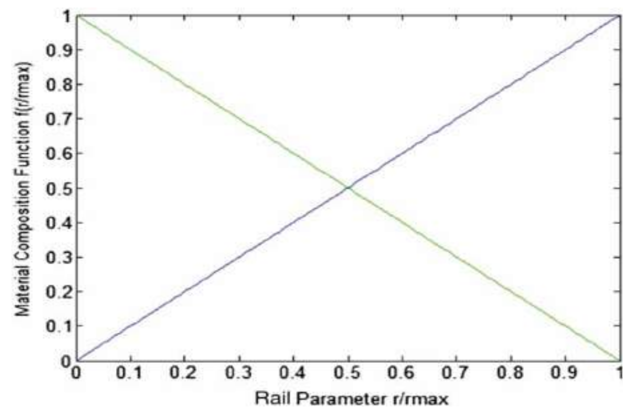


**Figure 5.** Linear Material Composition Function over the parameterized rail.

## 5. Hybrid representation scheme

The definition of the hybrid representation follows these steps:

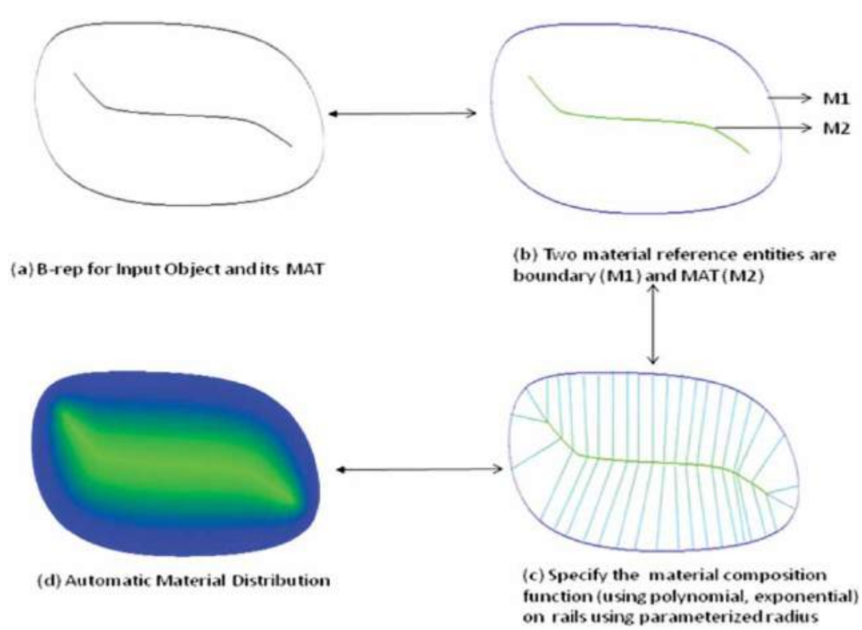


**Figure 4.** Schematic Process of Heterogeneous Object Modeling.

### 5.1. Specify the material reference entity

The geometry and the material composition of the material reference entity is specified. Material reference entities are the boundary and MAT. Both are represented using the graph based models (V, E, F) like Boundary representation. In the tuple (V, E, F), V is the set of vertices, E is the set of edges connecting these vertices, and F is the set of facet connecting these edges. For 2D, field F does not exist.

Geometry of rails (g) joining MAT to foot-point is represented using its end points.

$$g = (P, S) \tag{2}$$

where P and S are the coordinates of the foot-point and corresponding MAT point.

If material composition vector $(P_M, S_M)$ of these point sets (foot-point and MAT points) are coupled with their geometric coordinates$(P, S)$, the rail can be represented as

$$g = (P, S, P_M, S_M) \tag{3}$$

The material composition vector associated with the foot-points and MAT-points can be assigned either manually or using existing library of functions [23].

The set of rails can be stored using $G^d$, where d is the dimensionality of the point-set, and $n_R$is the total number of rails.

$$G^d = \{g_k\}, \quad k \in [1, n_R] \tag{4}$$

In the example in Figure 6, d = 2 and number of rails $n_R$ is 40. The representation is defined at a specified number of rails as this is an outcome of the way the medial axis is represented as a set of points. As the material function is available along the domain boundary and the medial axis, the representation for the rails can be constructed once the number of rails is specified.

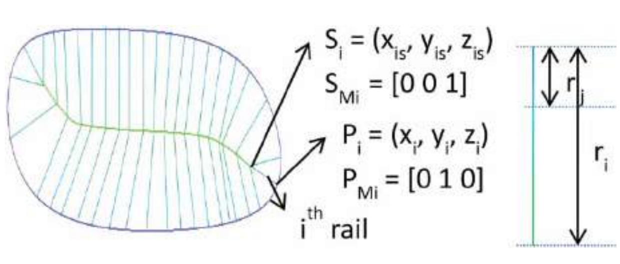


**Figure 6.** (a) Geometric coordinate coupled with the material coordinate for i$^{th}$ rail. (b) The rail parameter for any point at j$^{th}$ index on i$^{th}$ rail is given by $r_{ij}/r_i$.

### 5.2. Specify the material distribution function along a rail

The material composition in the object is controlled by specifying the material gradient function between the MAT and the boundary along a rail.

*Radial Distribution function:* This function can be defined over the rail using polynomial, exponential or any other distance function of the rail parameter. The material composition at the start and end is already known on each rail. The rail parameter for any point on i$^{th}$ rail is given by $r_{ij}/r_i$ where $r_{ij}$ is the distance of the point from the foot-point on i$^{th}$ rail. It has to be noted that the start point of the rail is the foot-point and end-point of the rail is corresponding MAT point.

A linear material function is given by $f\left(\frac{r_{ij}}{r_i}\right) = \frac{r_{ij}}{r_i}$.

$$P_{Mij} = P_{Mi} + f\left(\frac{r_{ij}}{r_i}\right).(S_{Mi} - P_{Mi}) \tag{5}$$

Here, $P_{Mij}$ is the material composition for the j$^{th}$ point on i$^{th}$ rail, $S_{Mi}$ and $P_{Mi}$ are material composition at the MAT point and corresponding foot-point respectively of i$^{th}$ rail.

Non-linear distribution functions can be defined for higher degree (degree 2 and 3 is shown below).

$$f\left(\frac{r_{ij}}{r_i}\right) = \left(2.\left(\frac{r_{ij}^2}{r_i^2}\right) - 3.\left(\frac{r_{ij}}{r_i}\right) + 1\right) \tag{6}$$

$$f\left(\frac{r_{ij}}{r_i}\right) = \left(2.\left(\frac{r_{ij}^3}{r_i^3}\right) - 3.\left(\frac{r_{ij}^2}{r_i^2}\right) + 1\right) \tag{7}$$

The material distribution function is stored with the rails in field A.

*Method to evaluate Material Composition at any point Q:* For any point Q inside the domain in Figure 7 (a), the material composition can be evaluated from the prescribed material composition function by plugging in value of the rail parameter $r_{ij}/r_i$ for point Q. Evaluation of this parameter needs the start and end point of the rail through Q. Let the material composition at the rail through Q be $Q_{Ms}$ and $Q_{Mk}$ at the MAT point and corresponding foot-point respectively. The rail parameter is given by the ratio of the distance between $QQ_k$ to $Q_sQ_k$ , as shown in Figure 7 (b).

Thus, material composition is evaluated using the given material distribution function as follows:

$$Q_M = Q_{Mk} + f\left(\frac{distance(QQ_k)}{distance(Q_sQ_k)}\right).(Q_{Ms} - Q_{Mk}) \tag{8}$$

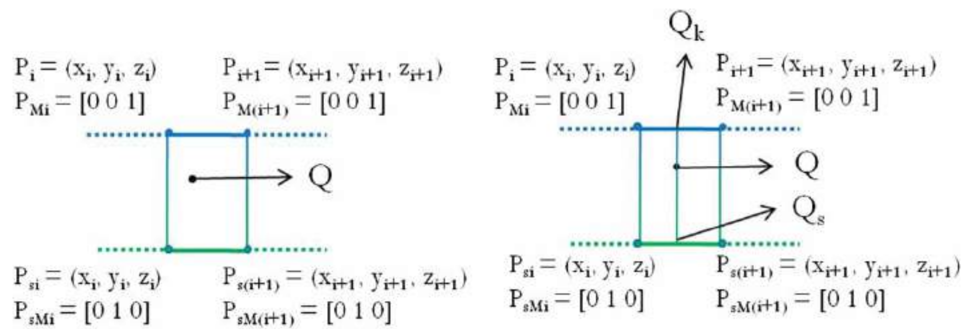


**Figure 7.** (a) Query point Q inside the track (b) Insertion of the new rail $Q_sQ_k$ through Q.

It is to be noted that the rail through Q is evaluated only for the purpose of material interrogation. It is not stored in the representation.

***A pseudo-code description of the material interrogation in the radial direction is stated as follows.***

Algorithm: Material Interrogation for any point Q (x, y, z)

Input: Heterogeneous Object Model (G, A)

Output: Material Composition vector $[m_1, m_2, m_3 \ldots\ldots m_n]$

i. Find the MAT point closest to the point Q(x, y, z). Corresponding to this MAT point, find the cell C containing this point Q.
ii. Find the foot-point $Q_k$ on cell C by projecting Q on its point/edge/facet formed by only foot-points as vertices.
iii. Find the corresponding MAT point $Q_s$ on cell C by intersecting the line joining $Q_kQ$ on its point/edge/facet formed by MAT points as vertices.
iv. Evaluate the barycentric coordinate of the point $Q_k$ and $Q_s$ wrt the vetex/edge/facet on which it lies.
v. Using the barycentric coordinates, evaluate material composition vector $Q_{Mk}$ and $Q_{Ms}$.
vi. Evaluate the rail parameter for the new rail by using the ratio of the $r_{ij}/r_i = distance(QQ_k)/distance\ (Q_kQ_s)$.
vii. Substitute this parameter in the material distribution function as $Q_M = Q_{Mk} + f\left(\frac{r_{ij}}{r_i}\right).(Q_{Ms} - Q_{Mk})$.
viii. return $Q_M$.

***Algorithmic Complexity:*** The algorithmic complexity is determined by the search for the closest MAT point which is O (log n), where n is the number of MAT points.

Above two steps (5.1 and 5.2) have shown that the rails are stored using $G^d$ and radial material distribution functions are stored using A.

Thus, for an HOM of Object O, representation can be written as follows:

$$O = (G, A) \tag{9}$$

## 6. Evaluated form of hybrid representation

The evaluated form of the material distribution is a discretization of the domain into elements that is desired for finite-element analysis, manufacturing and visualization. This step approximates the prescribed material distribution on the rails (from field A in the representation) by inserting nodes on the rail. While higher order approximations could be used, linear approximation is chosen as they can be used directly in applications such as rendering, tool path generation and linear Finite element method(FEM). The level of approximation can be controlled by prescribing error tolerance. The resulting discretization on each rail is stored using third field N. The evaluated representation is given by (G, A, N). Nodes on two adjacent rails are connected in the sequential order starting from the foot-point to stopping at MAT point. In case two adjacent rails contain unequal number of nodes, the remaining nodes on one rail, after matching all other nodes, are connected to the last terminating node (MAT point) of the other rail.

*Adaptive Subdivision along rail:* As each element uses linear interpolation for material distribution within each element, it is important to covert the given material distribution function on the rail into piece-wise linear approximation by inserting nodes on the rail. This step describes the procedure that uses first order approximation of the material distribution function.

Let the material distribution function be $f(r_{ij}/r_i)$ and its first order approximation be $\overline{f(r_{ij}/r_i)}$. Using Taylor series, the first order approximation can be written as follows:

$$\overline{f\left(\frac{r_{ij}}{r_i}\right)} = f\left(\frac{r_{i(j-1)}}{r_i}\right) + f'\left(\frac{r_{i(j-1)}}{r_i}\right).t \tag{10}$$

Here, t is the step size along the rail i.e. $t_{ij} = r_{ij} - r_{i(j-1)}$.

For an acceptable error tolerance *tol* between exact function and the approximating function, the following relation should be respected.

$$\left| f\left(\frac{r_{ij}}{r_i}\right) - \overline{f\left(\frac{r_{ij}}{r_i}\right)} \right| < tol \tag{11}$$

Solving the equation for t for maximum tolerance *tol*, parametric location of the nodes to be inserted on the rail is evaluated as $r_{ij} = r_{i(j-1)} + t$ where j = 1, 2, 3 . . . . ..

Thus, nodes on the $i^{th}$ rail can be stored using $N_i$.

$$N_i = \left\{\frac{r_{ij}}{r_i}\right\}_{j=0}^{J} \tag{12}$$

Iterating over all the rails, it can be generalized as follows:

$$N = \{N_i\}_{i=1}^{I} = \left\{\left\{\frac{r_{ij}}{r_i}\right\}_{j=0}^{J}\right\}_{i=0}^{I} \tag{13}$$

Figure 8(a) illustrates the approximation of quadratic material distribution function, i.e. $\left(2 \cdot \left(\frac{r_{ij}^2}{r_i^2}\right) - 3 \cdot \left(\frac{r_{ij}}{r_i}\right) + 1\right)$. The nodes are inserted on the rail at the location where the deviation equals tolerance (set to be 0.1). The exact distribution is shown in red while the deviation is shown in blue. Figure 8 (b) shows the inserted nodes on the rail.

After inserting the nodes, the resulting material distribution function, which is piece-wise linear, is shown in Figure 9 (a). The exact function is in red color,

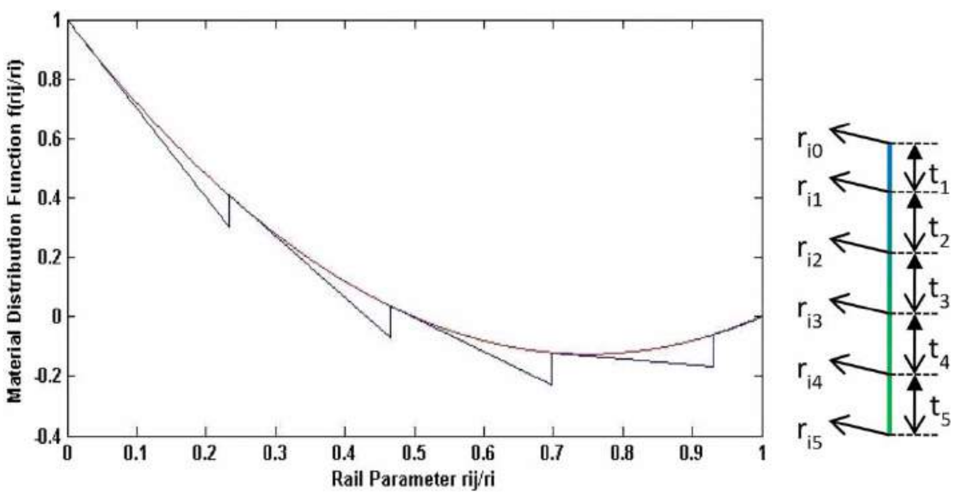


**Figure 8.** (a) Linear approximation of the material distribution function along the rail (b) Discretization of the rail for the different approximating interval t1, t2, t3, t4, t5.

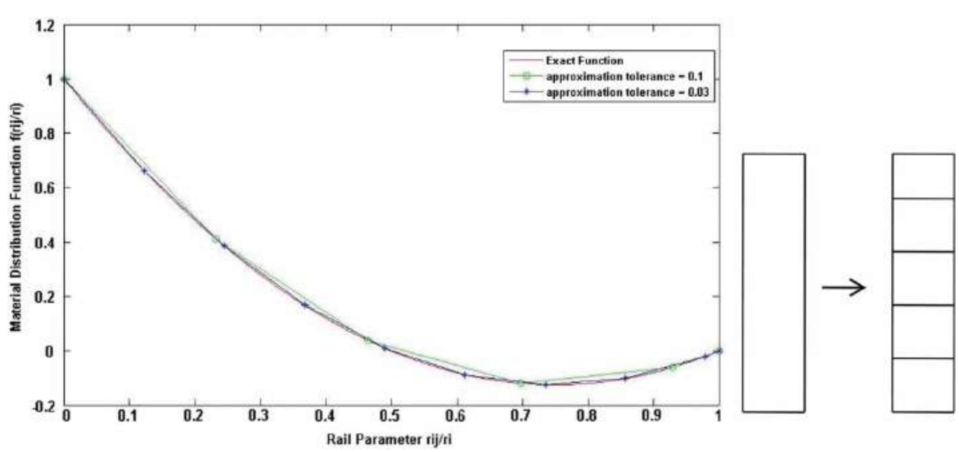


**Figure 9.** (a) Adaptive subdivision of rail to approximate the given material distribution (b) Discretization of track for the corresponding discretization on the rail.

approximating function with tolerance 0.1 and 0.03 are shown in green and blue respectively.

It has to be noted that while discretizing the given material distribution, second field A may be used to store the given material distribution function. But to evaluate material composition at any point on the rail, the rail parameter at that point has to be calculated from the given reference entity and then plugged in to the given material distribution function. So, it is better to store the material composition vector at the each rail parameter of the nodes explicitly.

$$A = \{A_i\}_{i=1}^{I} = \{\{P_{Mij}\}_{j=0}^{J}\}_{i=0}^{I} \tag{14}$$

where $P_{Mij}$ is the material composition vector of the node at j$^{th}$ parameter on i$^{th}$ rail.

*Material distribution within each element:* The nodes on the rail due to discretization of the function are matched with the adjacent rail to generate elements within the track as shown in Figure 9 (b). Material distribution within each element is equivalent to finding the material composition at all interior points of the element, which can be achieved by using procedure for material interrogation for a given point described under section 5.2. However, most of the tools used for rendering, analysis and manufacturing support vertex based interpolation like barycentric or bilinear interpolations, which are described below.

(a) *Barycentric Interpolation:* Barycentric interpolation linearly interpolates the material composition within an element using the linear combination of the material composition on its vertices. Any point Q inside the element is given as follows:

$$Q_M = C_1P_{M1} + C_2P_{M2} + C_3P_{M3} + C_4P_M \tag{15}$$

where $C_1, C_2, C_3$ and $C_4$ are barycentric coordinates, $C_1 + C_2 + C_3 + C_4 = 1$, $C_1 \geq 0$, $C_2 \geq 0, C_3 \geq 0$, $C_4 \geq 0$; $P_{M1}$, $P_{M2}$, $P_{M3}$ and $P_{M4}$ are material composition vectors at the respective vertices of the elements.

(b) *Bilinear Interpolation:* Let any point Q have parametric value u and v inside the element. It first linearly approximates the composition in one direction using u and then linearly approximates in other direction using v.

$$\begin{aligned} Q(u, w) = {} & P_{M1}(1-u)(1-w) + P_{M2}(1-u)\,w \\ & + P_{M3}u \times (1-w) + P_{M4}u\,w, \\ & \times 0 \leq u \leq 1, 0 \leq v \leq 1 \end{aligned} \tag{16}$$

This method distributes the material linearly only on the bounding edges while at other points inside the element, it has quadratic distribution. In this paper, all the results are rendered using barycentric interpolation.

Thus, hybrid representation can be discretized for the given material distribution function. The evaluated form can be further subdivided to any level of resolution without losing the desired accuracy, which is independent of the input point sampling just like unevaluated model.

## 7. Editing the HOM

The material distribution can be interactively varied without changing the geometry. The material distribution can be varied in two ways: first is by changing the material composition vector at the start and end of the rails; second, by changing the material distribution function over the rail. For these changes, the representation needs to be updated.

For the first change, $P_{Mi}$ and $S_{Mi}$ alter in the first field of the representation, while for the second change, material function changes in the second field A.

Figure 10 shows the alteration in material composition by shifting green to blue, and vice versa. Another change is shown in material gradation due to the change in degree of the material composition function from degree 1 to 3.

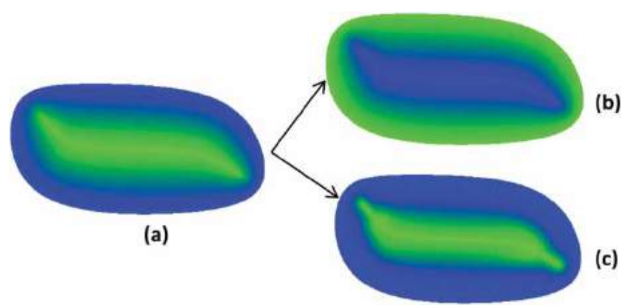


**Figure 10.** Alteration in material composition by swapping material composition at the start and end of the rails and changing the degree of the distribution function from 1 to 3.

## 8. Material continuity

The material continuity is required to avoid the conflicting properties like thermal expansion across the common boundary and to ensure better bonding of the microstructure. The continuity is ensured by using the interpolations schemes like barycentric or bilinear. If we take two adjacent tracks ($P_i$, $P_{i+1}$, $S_{i+1}$, $S_i$) and ($P_{i+1}$, $P_{i+2}$, $S_{i+2}$, $S_{i+1}$), it can be shown easily that under limiting conditions, the material composition of a point in these tracks converges to the same value at

**Figure 11.** Continuity in randomized material distribution.

the common edge. This capability enhances the potential of this method to model even random material distribution without sacrificing the material continuity. Figure 11 shows the continuous distribution for the points set given random material composition on MAT and boundary points.

## 9. Data structure

Data structure for Hybrid representation where material distribution is prescribed using rail stores two elements/attributes G and A. For hybrid representation discretizing the given material distribution, it stores all three elements G, A and N.

Figure 12 shows the data structure of elements G and A in the tuple. The geometric domain consists of the set of rails $\{g_i\}_{i=1}^{I}$. Each rail consists of foot-point and corresponding MAT point. Foot-point structure stores geometric coordinate and material coordinate (material composition vector) of the foot-point for a given rail identifier. Mat-point structure also stores the geometric coordinate and material coordinate of MAT point for a given rail identifier. Adjacent rails form a track, which is a simplex called cell. Cell structure stores the reference to foot-point and MAT point. In 2-D, cell may be a triangle or quadrilateral, while in 3-D, it may be tetrahedron, pyramid, wedge or hexahedron. The cell structure contains references to 3 MAT and 3 foot-points, but it can have any combination depending in the type of cell. For example, a tetrahedron may be formed by one foot-point and corresponding three MAT points. For tetrahedron, there are 3 rails, each connecting foot-point to its MAT point. Second element A stores the material distribution function for each rail.

Figure 13 shows the data structure of the hybrid representation. In this only the element A is modified by the introduction of new nodes. Second field A stores the set of material composition vector at the nodes on each rail. If the material composition function on each rail is different, it can stored as $f_i(r_{ij}/r_i)$. N stores the set of parameters $r_{i1}, \ \ r_{i2} \ldots .r_{iN}$ of the nodes on each rail.

To use the data structure for querying material composition efficiently, it is important to find the cell containing

<table>
<tr><th colspan="5">First Field G</th></tr>
<tr><th colspan="5">Foot-point structure</th></tr>
<tr><td>Rail id</td><td>Foot-point id</td><td>Geometric Coordinate</td><td>Material Coordinate</td><td>Cell id</td></tr>
<tr><td>$g_i$</td><td>$V_i$</td><td>$(x, y, z)_i$</td><td>$(m_1, m_2, m_3.....m_n)_i$</td><td>$C_1, C_2, C_3....$</td></tr>
<tr><td colspan="5"></td></tr>
<tr><th colspan="5">MAT-point structure</th></tr>
<tr><td>Rail id</td><td>MAT-point id</td><td>Geometric Coordinate</td><td>Material Coordinate</td><td>Cell id</td></tr>
<tr><td>$g_i$</td><td>$V_{si}$</td><td>$(x, y, z)_{si}$</td><td>$(m_1, m_2, m_3.....m_n)_{si}$</td><td>$C_1, C_2, C_3....$</td></tr>
<tr><td colspan="5"></td></tr>
<tr><th colspan="5">Cell structure</th></tr>
<tr><td>Cell id</td><td colspan="2">MAT-point id</td><td colspan="2">Foot-point id</td></tr>
<tr><td rowspan="3">$C_v$</td><td colspan="2">$V_{s1}$</td><td colspan="2">$V_1$</td></tr>
<tr><td colspan="2">$V_{s2}$</td><td colspan="2">$V_2$</td></tr>
<tr><td colspan="2">$V_{s3}$</td><td colspan="2">$V_3$</td></tr>
<tr><th colspan="5">Second Field A</th></tr>
<tr><td colspan="2">Rail id</td><td colspan="3">Material Composition Function</td></tr>
<tr><td colspan="2">$g_i$</td><td colspan="3">$f(r_{ij}/r_i)$</td></tr>
</table>

**Figure 12.** Data structure for the elements G and A for prescribed material distribution on rails.

| **Second Field A** | | **Third Field N** | |
|---|---|---|---|
| Rail id | Material Composition Function | Rail id | Rail Parameter |
| $g_i$ | $f(r_{ij}/r_i)$ or $P_{Mij}$ | $g_i$ | $r_{i1}, r_{i2}, r_{i3}$........... |

**Figure 13.** Data structure of second and third element A and N for hybrid representation.

the query point. The cells containing the query point has MAT point and foot-point as vertices. If the closest MAT point or foot-point to a query point is known, then the cell containing it can be retrieved as each MAT or foot-point maintains the reference to the cells in its structure. Thus, the only expensive process is finding the closest MAT or foot-point to the query point. If MAT point is stored using K-d trees [1], the search for closest MAT point has logarithmic complexity O(log n), where n is the number of MAT points.

## 10. Results and discussion

The hybrid approach has been implemented in Windows 7 using Microsoft Visual Studio 2008. The input models of the domain are taken as .sat files [16]. The elements are obtained using LayTracks algorithm [18]. The outputs are rendered using openGL [13] and VTK [21].

Figure 14 shows a dental implant exhibiting different material properties. The exposed portion of the implant has to be more wear resistant and hard, while the root portion has to be bio-compatible. The material gradation is controlled from inside to outside to bring necessary hardness and toughness to avoid any crack propagation or fatigue. Four different material reference entities and linear material composition functions are used. First material composition is assigned to the exposed portion (for hardness and wear), second material composition is assigned to the rooted portion (for bio-compatibility). Third material composition is assigned to the MAT segments (for controlling toughness from inside to exposed boundary) corresponding to the exposed portion, while the fourth material composition is assigned to the MAT segments belonging to the root portion (for controlling toughness from inside to rooted boundary).

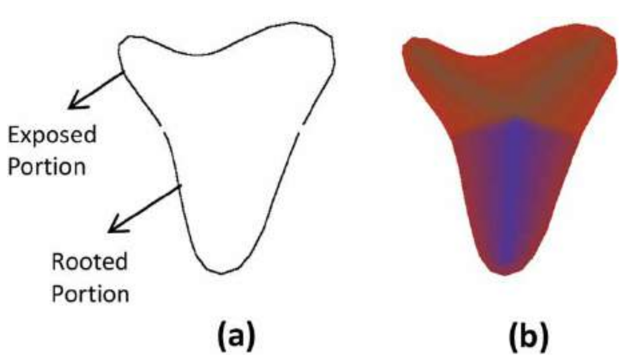


**Figure 14.** Dental Implant.

Figure 15 shows blade consisting of metal in yellow and ceramic in red. Metal gives the mechanical strength and toughness while ceramic gives high wear, heat and chemical strength. The material composition gradient between metal and ceramic can be controlled from inside to outside to avoid conflicting material properties or vary thickness of the ceramic coating as shown in Figure 15 (a) and (b).

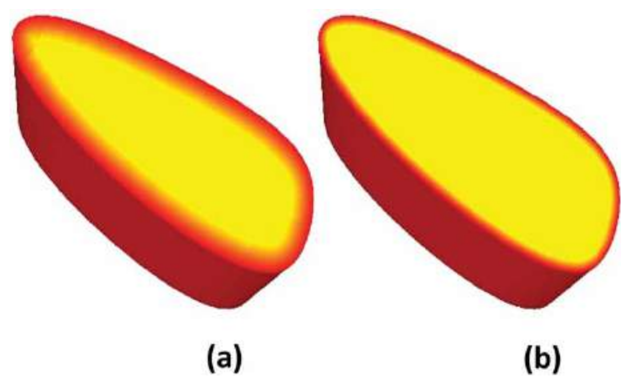


**Figure 15.** (a) and (b) shows two-material blade with different material composition variations from inside to outside.

Figure 16 shows the HOM model of bone where the outer boundary is given material composition of hard material, and MAT is given material composition of the tough material, and the gradation from inside to out is applied to achieve functionally graded properties.

Figure 17 shows high stiff material in red and low stiff material in blue and transition between them. The overall stiffness decreases from bottom to top, which can be used in applications to optimized weight and cost where bottom is fixed and top is free.

MAT can be generated at the complexity of O (m log m) [22], where m is the number of points on the boundary. The algorithmic complexity for material interrogation is O(log n), where n is the number points on MAT.

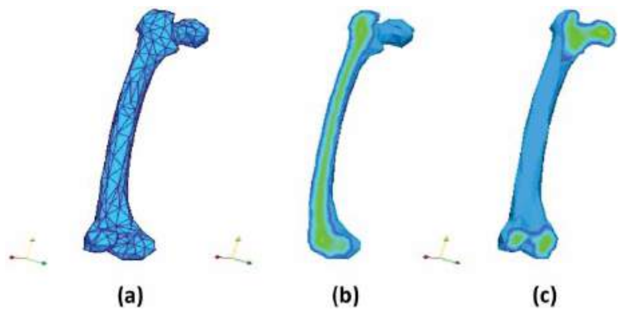


**Figure 16.** (a) 3-D HOM of bone (b) and (c) show sliced model.

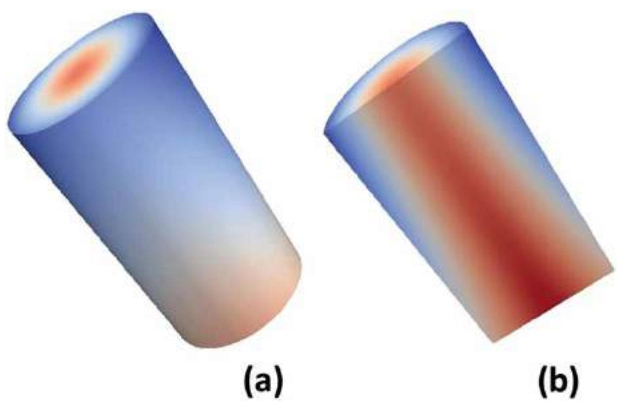


**Figure 17.** Heterogeneous stem with the decrease in stiffness from bottom to top in (a) and sliced model in (b).

The method allows the change in material function without any change in geometry without any computational effort. However, the change in geometry leads to the regeneration of MAT, which is of algorithmic complexity of O (m log m).

The existence of MAT for all objects enables the hybrid representation to model wide range of heterogeneous objects like evaluated model and the decomposition of the geometry can be done at any level resolution like unevaluated model.

In this paper, material function is defined using the rail parameter, which is normalized using the maximum length of the corresponding rail. However, length of each rail can be normalized by the global maximum length that will be the representative of the girth of the object. The material variation using girth or rail parameter of the object is useful to develop the multi-material (or multi-functional) devices to mimic the functions of naturally occurring objects like bone, bamboo calm where hardness decreases from outside to inside optimizing the structural weight. Aerospace applications, turbine blades etc can also benefit in optimizing the weight with the use of different material functions at different radial locations (like wear resistance material like ceramic at the outside, metal in the intermediate layer for strength and voids in the interior to enhance the mechanical advantage against bending or torsion).

## 11. Conclusion

This paper has presented a hybrid representation to model heterogeneous objects, which overcome the limitation of the existing evaluated and unevaluated model. The idea was to decompose the geometry such that existing class of material functions can be defined to achieve the desired material variation in heterogeneous objects. Using MAT, the geometry can be decomposed into region outside to inside. The representation has been built using the continuous representation of MAT of the object and materials functions defined on rail of MAT that has well-defined start and termination points. The resulting distribution has been shown symmetric, intuitive to vary from inside to outside, efficient to evaluate material composition at a given location in the object and could be adaptively discretized in the direction of material gradient without much computational effort. The future work is to extend the representation for the general heterogeneous objects where the material distribution can be because of the random and irregular material features.

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