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Reformulation of Generalized Log-aesthetic Curves with Bernoulli equations

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ABSTRACT
A recent research has found out the relationship between the Log-aesthetic Curve and Riccati differential equations. The obtained differential equations are Riccati, but also Bernoulli type. Based on Sato and Shimizu's derivation of the formula of ρ-shift generalized log-aesthetic curve (GLAC), we derive the formula of the κ-shift GLAC as a solution of a Bernoulli differential equation.

KEYWORDS
Log-aesthetic curve; Generalized Log-aesthetic Curve; Bernoulli differential equations; Riccati differential equations

1. Introduction

A Generalized Log-aesthetic Curve (GLAC) is the general formulation of emerging the Log-aesthetic (LA) curves for aesthetic industrial design. GLAC has an extra degree of freedom compared to LA curve which makes it versatile for design. There are two approaches employed to develop GLACs namely ρ-shift and κ-shift [2]. To note, κ-shift GLAC is a better formulation of GLAC since its direction angle can be obtained analytically as compared to ρ-shift GLAC. Figure 1 shows two examples of κ-shift GLAC with a LAC, their radius of curvature (RoC) and logarithmic curvature graph (LCG) [2].

Recently, Sato and Shimizu [5] reported the relationship between the fundamental equation of Log-aesthetic curve and Riccati differential equations. They considered the case of ρ-shift GLAC and reported its representation in the form of Riccati equation. It is well known that solving Riccati equation involves reduction of order which is a painstaking trial and error approach to find for a solution. This paper completes the investigation by analyzing κ-shift GLAC. We derived the formula of the κ-shift GLAC as a solution of a Bernoulli equation which can be solved with various approaches.

2. Generalized cornu spiral

In this section the formulation of the generalized Cornu spiral is explained and its relationships with the log-aesthetic curve and the generalized log-aesthetic curves are discussed.

2.1. Formalization and its properties

The generalized Cornu spiral (GCS) [3] is defined as a curve whose curvature profile is given by a linear rational function and when the arc length of the curve is assumed to be s, the curvature κ of the curve is given by

$$\kappa(s) = \frac{p+qs}{S+rs} \tag{2.1}$$

where p, q, $r > -1$ and $S > 0$ are constants. In case of $q = 0$ or $r = 0$, the curve is a logarithmic spiral or a clothoid curve, respectively. Hence the GCS is considered to be a generalized curve of the clothoid curve as well as the logarithmic spiral. The differentiation of the above equation with respect to s yields

$$\frac{d\kappa(s)}{ds} = \frac{Sq-pr}{(S+rs)^2}. \tag{2.2}$$

The domain of definition of the curve is $0 \leq s \leq S$. Since $d\kappa(s)/ds$ does not change its sign for the whole domain, the curvature increases or decreases monotonically.

Assume that the curvatures at the start and end points are given by $\kappa_0 = p/S$ and $\kappa_1 = (p+qS)/(1+r)S$, respectively. Since the curvature $\kappa(s)$ is a derivative of the direction angle $\theta(s)$, $\theta(s)$ is given by

$$\frac{dy}{dx}\theta(s) = \theta_0 + \frac{pr-Sq}{r^2}\log\left(1+\frac{rs}{S}\right) + \frac{qs}{r} \tag{2.3}$$

where θ_0 gives the direction angle at the start point. The term *log* corresponds to the variation of the direction angle of a clothoid curve. The term qs/r increases in

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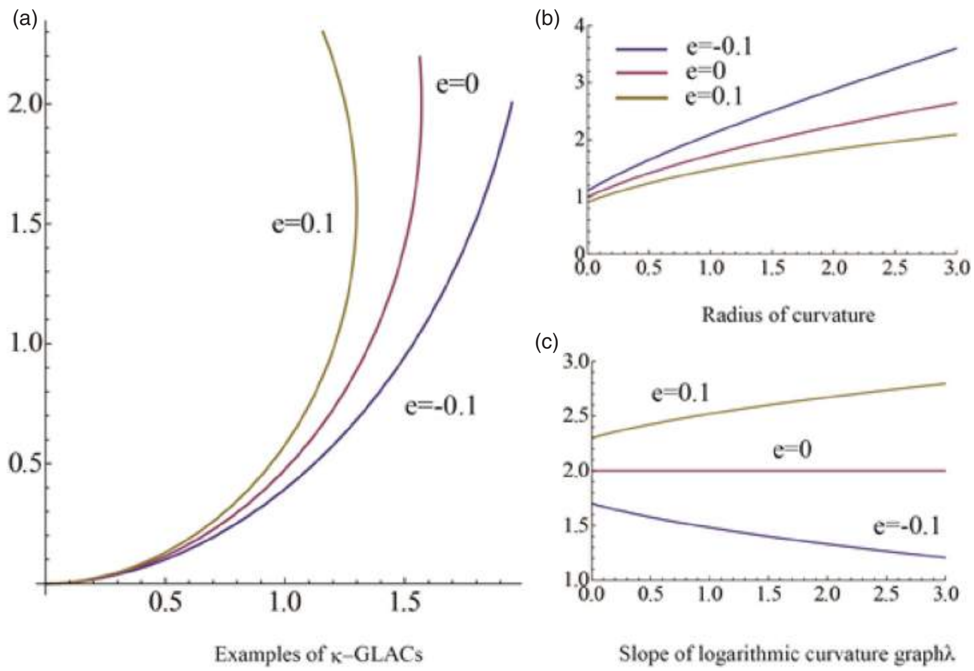

Figure 1. Examples of -shift GLAC and their RoC and LCG.

proportion to s and as a result it increases the direction angle. It corresponds to the variation of the direction angle of a circular arc. It can be said that the variation of the direction angle of the GCS is a combination of those of the clothoid curve and the circular arc.

2.2. Comparison with log-aesthetic curve and its generalization

To introduce α into the GCS, we assume that the slope of the logarithmic curvature graph is expressed by λ in the following discussions. Gobithaasan and Miura [2] showed that λ of an arbitrary planar curve is given by

$$\lambda(t) = 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho'(t)s''(t)}{s'(t)} - \rho''(t)\right) \tag{2.4}$$

When the curve is parametrized by the arc length s, it is given by

$$\lambda(s) = 1 - \frac{\rho(s)\rho''(s)}{\rho'(s)^2} \tag{2.5}$$

Hence λ of the GCS is given by

$$\lambda(s) = \frac{2r(-\kappa_0 + \kappa_1 + r\kappa_1)}{(1+r)S(\kappa_0 - \kappa_1)}s + \frac{2r\kappa_0}{(1+r)(\kappa_0 - \kappa_1)} - 1 \tag{2.6}$$

The above expression indicates that $\lambda(s)$ is given by a linear function of the arc length s.

From Eq. (2.1) the curvature is given by

$$\rho(s) = \frac{S + rs}{p + qs} \tag{2.7}$$

It is possible to extend the above equation as follows:

$$\rho(s)^\alpha = \frac{S + rs}{p + qs} \tag{2.8}$$

similar to that of the log-aesthetic curve. However, it explicitly includes the total length S of the curve. The value of the constant r is restricted and it should be greater than −1. When the signs of p and q are different, the sign of the right expression could change its sign because of the change of the sign of its denominator $p + qs$. There are four parameters (p, q, r and S) and the expression of the direction angle obtained by the integration of the curvature includes a hypergeometric function. Therefore we generalize Eq. (2.7) as follows.

$$\rho(s) = \frac{S + rs}{p + qs} = f + \frac{g}{p + qs} \tag{2.9}$$

where $f = r/q$ and $g = S–pr/q$ are constants. Hence it can be transformed into

$$(\rho(s) - f)^{-1} = cs + d \tag{2.10}$$

where $c = q/g$, $d = p/g$. This equation means that $\rho(s)$ shifted by f is given by a linear function of s. α can be introduced to this equation as follows:

$$\rho(s) = (cs + d)^{\frac{1}{\alpha}} + f \tag{2.11}$$

The right side of the above equation has three parameters c, d and f if we exclude α. The meaning of the equation can clearly be interpreted as the radius of curvature is shifted by f. This type of the generalization is called ρ-shift.

It is possible to start from Eq. (2.1) and to perform the same extension for the curvature $\kappa(s)$ as follows:

$$\kappa(s) = (cs + d)^{-\frac{1}{\alpha}} + f \tag{2.12}$$

This extension has several practical merits and this type is called κ-shift. The curve obtained by either of these two types of generalization is the Generalized Log-Aesthetic Curve: GLAC.

3. Riccati differential equations

Generally a Riccati differential equation [1] is given by

$$\frac{dy}{dx} + P(x) + Q(x)\ y + R(x)y^2 = 0 \tag{3.1}$$

It is known that equation (3.1) cannot be solved by integration in general. However if a particular solution is known, then it can be solved. This solution method is known as reduction of order which may involve trial-error approach. In the case of $P(x) = 0$, we may convert to Bernoulli equation in which a feasible solution can be obtained by various means. Assume that a particular solution of Eq. (1) is given by $\eta(x)$ and the general solution is given by $\mathrm{y} = \eta(\mathrm{x}) + z(x)$. Then Eq. (3.1) becomes

$$\frac{dz}{dx} + \{Q(x) + 2R(x)Q(x)\ \eta\}z + R(x)z^2 = 0 \tag{3.2}$$

The above equation is classified as a Bernoulli equation which is discussed in the following section.

4. Bernoulli differential equations

Let n be a constant and $n \neq 1$, a Bernoulli differential equation [1] is given by

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{4.1}$$

If $n = 2$, equation (4.1) is regarded as a Riccati equation as shown in the previous section. By dividing both sides of Eq. (4.1) with y^n, we obtain

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x) \tag{4.2}$$

By letting $z = 1/y^{n-1}$, we can further simplify as follows

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x) \tag{4.3}$$

The equation (4.3) is a linear differential equation which can be solved by separation of variables.

5. Riccati equations satisfied by the log-aesthetic curve

According to Sato and Shimizu [5], similarity curvature is defined as follows:

$$S := \frac{\frac{d\kappa}{ds}}{\kappa^2} = -\frac{d\rho}{ds} \tag{5.1}$$

where s is arc length, κ is curvature and $\rho = 1/\kappa$ is radius of curvature. The above curvature is invariant under similarity transformation. Since the arc length s is variant by similarity transformation, we may use the direction angle θ to define the curve $\gamma(\theta)$. Since $ds = \rho d\theta$, the similarity curvature is given by

$$S(\theta) = -\frac{\frac{d\rho}{d\theta}}{\rho} \tag{5.2}$$

A curve is said to be a log-aesthetic curve (LAC) if it has a linear logarithmic curvature graph (LCG). Thus, in 2006 Miura [4] developed the fundamental equation of LA curves by extracting curvature function from a linear LGC. The Y axis of the logarithmic curvature graph (LCG) is given by

$$Y = -\log\left(\left|\frac{d(\log\rho)}{ds}\right|\right) = X - \log(|S|) \tag{5.3}$$

where $X = \log\rho$. The gradient (α) of the logarithmic curvature graph is given by

$$\alpha = \frac{dY}{dX} = 1 - \frac{\frac{dS}{dX}}{S} = 1 + \frac{\frac{ds}{d\theta}}{S^2} \tag{5.4}$$

Therefore a LAC with shape parameter α satisfies the following Riccati equation:

$$\frac{dS}{d\theta} = (\alpha - 1)S^2 \tag{5.5}$$

However, we may convert it into a Bernoulli equation which can be solved with separation of variables. Its general solution is given by

$$S(\theta) = -\frac{\lambda}{(\alpha-1)\lambda\theta + 1} \tag{5.6}$$

We denote the right side of the above equation as $L(\alpha, \lambda; \theta)$. By solving this equation for $\rho(\theta)$, we obtain

$$\rho(\theta) = \begin{cases} e^{\lambda\theta} & (\alpha = 1) \\ ((\alpha-1)\lambda\theta + 1)^{\frac{1}{\alpha-1}} & (\alpha \neq 1) \end{cases}$$

6. The evolutes of log-aesthetic curves

For a given smooth curve $\gamma(s)$, its evolute $\sigma(s)$ is defined by

$$\sigma(s) = \gamma(s) + \rho_\gamma(s)\, n_\gamma(s) \tag{6.1}$$

where $\rho_\gamma(s)$ and $n_\gamma(s)$ are its radius of curvature and normal vector of curve $\gamma(s)$ respectively. Conversely $\gamma(s)$ is called an involute of $\sigma(s)$. The first derivative of $\sigma(s)$ is given by

$$\frac{d\sigma}{ds} = \frac{d\rho_\lambda}{ds} n_\gamma \tag{6.2}$$

The above equation indicates that the direction angle of curve $\sigma(s)$ is aligned along the normal vector of $\gamma(s)$ and its directional angle is rotated counterclockwise by 90 degrees. Note that from the above equation, parameter s is not an arc length parameter of $\sigma(s)$. Its second derivative is given by

$$\frac{d^2\sigma}{ds^2} = \frac{d^2\rho_\lambda}{ds^2} n_\gamma - \frac{1}{\rho_\gamma}\frac{d\rho_\gamma}{ds} T_\gamma \tag{6.3}$$

Hence

$$\frac{d\sigma}{ds} \times \frac{d^2\sigma}{ds^2} = \frac{1}{\rho_\gamma}\left(\frac{d\rho_\gamma}{ds}\right)^2 \tag{6.4}$$

Therefore the radius of curvature $\rho_\sigma(s)$ of curve $\sigma(s)$ is given by

$$\rho_\sigma(s) = \frac{\left|\frac{d\sigma}{ds}\right|^3}{\left|\frac{d\sigma}{ds} \times \frac{d^2\sigma}{ds^2}\right|} = \rho_\gamma \frac{d\rho_\gamma}{ds} = \frac{d\rho_\gamma}{d\theta} \tag{6.5}$$

Since $dS/d\theta = (d\rho_\gamma/d\theta)^2/\rho_\gamma^2 - (d^2\rho_\gamma/d\theta^2)/\rho_\gamma$, the relationship between similarity curvatures $S(\theta)$ and $T(\theta + \pi/2)$ is given by

$$\frac{dS(\theta)}{d\theta} = S(\theta)^2 - T\left(\theta + \frac{\pi}{2}\right) S(\theta) \tag{6.6}$$

We regard the above equation as a Bernoulli (special case of Riccati) equation which has $T(\theta + \pi/2)$ as a coefficient. When $\alpha \neq 1$, the similarity curvature $S(\theta) = L(\alpha, \lambda; \theta)$ of LAC can be rewritten as follows:

$$\frac{dS}{d\theta} = (\alpha - 1)S^2 = S^2 - \{(2-\alpha)S\}S \tag{6.7}$$

This equation is equivalent to the Riccati equation of the involute curve whose evolute's similarity curvature is given by $T(\theta + \pi/2) = (2-\alpha)L(\alpha, \lambda; \theta)$. Hence if $\alpha = 2$, the evolute of the LAC with $S(\theta) = L(\alpha, \lambda; \theta)$ is a circular arc ($T(\theta) = 0$). If $\alpha \neq \{1, 2\}$, the similarity curvature of the evolute is

$$\begin{aligned} T(\theta) &= (2-\alpha)L\left(\alpha, \lambda; \theta - \frac{\pi}{2}\right) \\ &= L\left(\frac{1}{2-\alpha}, (2-\alpha)\lambda; \theta - \frac{\pi}{2}\right) \end{aligned} \tag{6.8}$$

These results conform to those obtained by Yoshida and Saito [6].

Conversely we assume that $\alpha \neq \{1, 2\}$ and $T(\theta) = L\left(\frac{1}{2-\alpha}, (2-\alpha)\lambda; \theta - \frac{\pi}{2}\right)$. Then LAC is a particular solution of the following Riccati equation:

$$\frac{dS}{d\theta} = S(\theta)^2 - T\left(\theta + \frac{\pi}{2}\right) S(\theta) \tag{6.9}$$

The general solution is obtained using the above particular solution as follows

$$S(\theta) = \frac{L(\alpha, \lambda; \theta)}{1 + C((\alpha - 1)\lambda\theta + 1)^{\frac{1}{1-\alpha}}} \tag{6.10}$$

where C is a constant of integration. This gives the similarity curvature of the ρ-shift GLAC [2] as shown in the next section.

7. Similarity curvature of ρ-shift GLAC

The radius of curvature (ρ) of ρ-shift GLAC is given by

$$\rho_{\rho-GLAC}(\theta) = \begin{cases} e^{\lambda\theta} + \nu & (\alpha = 1) \\ ((\alpha\kappa - 1)\lambda\theta + 1)^{\frac{1}{\alpha-1}} + \nu & (\alpha \neq 1) \end{cases} \tag{7.1}$$

When $\alpha \neq \{1, 2\}$, its similarity curvature is given by

$$S_{\rho-GLAC} = -\frac{\frac{d\rho_{\rho-GLAC}}{d\theta}}{\rho_{\rho-GLAC}} = \frac{L(\alpha, \lambda; \theta)}{1 + \nu((\alpha - 1)\lambda\theta + 1)^{\frac{1}{1-\alpha}}} \tag{7.2}$$

The above equation is equivalent to Eq. (6.10).

8. κ-shift GLAC

As stated above, there are two types of GLAC [1]; ρ-shift and κ-shift. This section shows the derivation of κ-shift GLAC.

8.1. Reciprocal of similarity curvature

If the similarity curvature S is invariant under similarity transformation, its reciprocal $1/S$ is also invariant if $S \neq 0$. We define $V = 1/S$ and call it similarity radius of curvature since it is the reciprocal of similarity curvature.

As $dV/d\theta = -(dS/d\theta)/S^2$, Eq. (5.5) is written with V by

$$\frac{dV}{d\theta} = 1 - \alpha \tag{8.1}$$

The above differential equation can be solved analytically and we may obtain the following general solution:

$$V(\theta) = -\frac{(\alpha - 1)\lambda\theta + 1}{\lambda} \tag{8.2}$$

This satisfies $V(\theta) = 1/S(\theta)$ where $S(\theta)$ is given by Eq. (11). We denote this general solution as $M(\alpha, \lambda; \theta)$.

8.2. Derivation of κ-shift GLAC formula

Substituting $S(\theta)$ with $V(\theta)$, Eq. (6.6) becomes

$$\frac{dV(\theta)}{d\theta} = T\left(\theta + \frac{\pi}{2}\right) V(\theta) - 1 \tag{8.3}$$

The general solution of the above equation is given by

$$V(\theta) = -\frac{((\alpha - 1)\lambda\theta + 1)^{\frac{\alpha-2}{\alpha-1}} \left(C + ((\alpha - 1)\lambda\theta + 1)^{\frac{1}{\alpha-1}}\right)}{\lambda} \tag{8.4}$$

$$= \frac{1 + C((\alpha - 1)\lambda\theta + 1)^{\frac{1}{1-\alpha}}}{L(\alpha, \lambda; \theta)} \tag{8.5}$$

The above equation is indeed the reciprocal of $S(\theta)$ given in Eq. (7.2).

On the other hand, the similarity radius of curvature of the κ-shift GLAC whose curvature is equal to $((\alpha - 1)\lambda\theta + 1)^{\frac{1}{1-\alpha}} + \nu$ is given by

$$V_{\kappa-GALC}(\theta) = -\frac{((\alpha - 1)\lambda\theta + 1)^{\frac{\alpha}{\alpha-1}} \left(\upsilon + ((\alpha - 1)\lambda\theta + 1)^{\frac{1}{1-\alpha}}\right)}{\lambda} \tag{8.6}$$

Eqs. (8.4) and (8.6) may look different. However, if we define $\beta = 2–\alpha$, and substitute it into Eq.(8.6), we obtain

$$V(\theta) = -\frac{((1 - \beta)\lambda\theta + 1)^{\frac{\beta}{\beta-1}} \left(C + ((1 - \beta)\lambda\theta + 1)^{\frac{1}{1-\beta}}\right)}{\lambda} \tag{8.7}$$

Hence

$$V(-\theta) = -\frac{((\beta - 1)\lambda\theta + 1)^{\frac{\beta}{\beta-1}} \left(C + ((\beta - 1)\lambda\theta + 1)^{\frac{1}{1-\beta}}\right)}{\lambda}$$
$$= M(\beta, \lambda; -\theta) \left(1 + \nu((1 - \beta)\lambda\theta + 1)^{\frac{1}{\beta-1}}\right) \tag{8.8}$$

Therefore the solutions represent a κ-shift GLAC whose LCG gradient equals to $\beta = 2–\alpha$ and the curve direction is reversed. As expected, κ-shift GLAC reduces to $V(\theta) = M(\beta, \lambda; \theta)$ when $\nu = 0$.

9. Conclusions

Sato and Shimizu have shown the relationship between LAC and ρ-shift GLAC using the concept of similarity geometry and Riccati equation. In this paper, we complete the research rewriting the fundamental formula of κ-shift GLAC as a solution of a Bernoulli equation which complies with the results obtained in previous studies. One possible future research is that applying the same procedures to GLAC itself, we may be able to derive a new curve, which may extend its drawable region more.

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References

[1] Differential equations, University of Tokyo, 1960.
[2] Gobithaasan, R.U.; Miura, K.T.: Aesthetic spiral for design, Sains Malaysiana, 40(11), 2011, 1301–1305.
[3] Jamaludin, M.A.; Tookey, R.M.; Ball, J.V.; Ball, A.A.: The generalized cornu spiral and its application to span generation, Journal of Computational and Applied Mathematics, 102(1), 1999, 37–47. http://dx.doi.org/10.1016/S0377-0427(98)00207-6
[4] Miura, K.T.: A general equation of aesthetic curves and its self-affinity, Computer-Aided Design and Applications, 2006, 3(1–4), 457–464. http://dx.doi.org/10.1080/16864360.2006.10738484
[5] Sato, M.; Shimizu, Y.: The log-aesthetic curve and Riccati equations from the viewpoint of similarity geometry, Proc. JSIAM2014, 2014.
[6] Yoshida, N.; Saito, T.: Interactive aesthetic curve segments, The Visual Computer (Pacific Graphics), 22(9–11), 2006, 896–905. http://dx.doi.org/10.1007/s00371-006-0076-5