COMPUTER-AIDED DESIGN & APPLICATIONS, 2016
VOL. 13, NO. 1, 1–13
http://dx.doi.org/10.1080/16864360.2015.1059183

Reconfigurable snap-together sculpting

Carlo H. Séquin

University of California, Berkeley

ABSTRACT
A mixture of general-use and of some custom-designed plastic parts, fabricated on inexpensive layered manufacturing machines, is used to construct a variety of sculptural maquettes. This article describes the design and fabrication of a set of modular parts that permit the assembly of tubular sculptures as well as constructivist realizations of mathematical knots and links.

KEYWORDS
Tubular building blocks; "LEGO®-Knots"; fused-deposition modeling

1. Introduction

For many millennia, from the *Venus of Willendorf* [18] to the Greek and Roman marble statues, sculpting was a subtractive process; material was selectively removed from an original body of wood, stone, or ivory to free up a smaller, more artistic shape contained within.

During the last two centuries constructive sculpting techniques have come into their own, where individual pieces of material are assembled and held together with bolts, welds, string, or some kind of adhesive glue. This allowed the construction of more complex sculptures, made from different materials, and possibly containing movable parts.

In the last two decades a new revolution has taken place. The emergence of rapid prototyping machines based on layered manufacturing techniques permits the fabrication of extremely complex, partially hollow geometries that cannot be made with subtractive machining, because the inner parts of such shapes are not reachable by any existing machine tool. The fact that the price of such machines has dropped by two orders of magnitude in the last twenty years has made these machines available to the general public. A large audience can now create their own custom-made parts, either on their own inexpensive rapid-prototyping machines, or through an on-line service such as Shapeways [13]. This offers new possibilities also for artists.

Thanks to layered manufacturing, many more people can now experiment with various conceived geometries and quickly produce small maquettes for little costs and with fast turn-around times.

2. Modular and reconfigurable art

At the 1981 Design Automation Conference, in Nashville TN, Sculptor Frank Smullin [14] presented a reconfigurable sculpture *Fit to be Tied* (Fig. 1a). Nine obliquely cut tube segments were held together with bolted flanges. Initially they were laid out and connected as a single straight pipe lying on the ground. Subsequently the angled joints were reconnected with a 180° azimuth change, and the construction was transformed into a curled-up trefoil configuration. Other artists, such as Richard Zawitz [19] in his *Museum Tangle* (Fig. 1b) have used modular tubular elements to make sculptural forms that can be deformed smoothly and continuously.

The work reported in this article was originally inspired by two pieces of art work by Henk van Putten [17] (Fig. 1c,d) exhibited in the art exhibit of the 2013 Bridges conference in Enschede, Netherlands. The overall shape is based on some simple modules, generated by sweeping a square cross section along a circular arc [16]. Combining several such elements with different bending angles and sweep radii leads to intriguing geometrical sculptures, in which the modularity may not be immediately obvious. While these sculptures were designed based on a few geometrical modules, they were not really constructed from individual modular elements but were machined or fabricated as composite shapes.

After the 2013 Bridges conference an exploration was started to see whether it was practical to create some physical "snap-together" parts representing the key modular shapes, so that one could do real-time "hands-on sculpting" and compose many different shapes from

CONTACT C. H. Séquin sequin@cs.berkeley.edu

these parts in a matter of minutes. This led to project *"LEGO®-Knots"* presented at the 2014 Bridges conference in Seoul [12].

2.1. *Basic* Borsalino *geometry*

This exploratory effort started out with the design of the two parts needed for the construction of the *Borsalino* shape (Fig. 1c) as explained by Henk van Putten [16]. All parts are basically sweeps of a square cross section along sweep curves that form circular arcs. The *Borsalino* needs two building blocks (Fig. 2a): the three (orange) end-caps that form tight 180° turns, and the six (green and cyan) connector pieces, which exhibit gentler bends through an angle of 45°. To form the tight, smoothly connected *Borsalino* configuration, the sweep radius, r, of the end-cap has to be half the side, s, of the square cross section, and the bending radius, R, of the medial axis of the connector has to be $1+\sqrt{2}$ times larger. By solving the quadratic equation derived from Figure 2b, which is the enlarged upper right corner of Figure 2a, one obtains:

$$R^2 + R^2 = (r+R)^2; \quad \rightarrow R^2 - 2^*R^*r - r^2 = 0;$$
$$\rightarrow R = r + r^*\sqrt{2}$$

Assembling the nine fabricated plastic parts resulted in a rather faithful reproduction of the basic *Borsalino* shape (Fig. 2c).

3. Fabrication issues

A second important goal of this effort was to design these parts in such a manner that they can be built readily on inexpensive rapid prototyping machines, such as the Afinia_H479 3D Printers [1], which dispense only a single plastic material. The aim was to minimize material costs and build times, as well as any subsequent cleanup required; it mandated geometries that in at least one build- orientation require only a minimal amount of supporting scaffolding that can be removed easily.

In a first round of implementation, all parts were built as hollow pipe segments with a square cross section. The nominal sleeve dimension by which consecutive pipe modules fit together was chosen to be one inch, the initial wall-thickness was 70mils (1.75 mm), and the sleeve insertion depth was 0.2″ (5 mm). Building a straight tube segment in a flat, horizontal position, would be very inefficient; the whole tube would be filled with (grey) scaffolding material (Fig. 3a). Theoretically the tube could be balanced on one longitudinal edge (Fig. 3b). Because cantilevered surfaces sloping out at 45° can be built without scaffolding, this build orientation may not require any

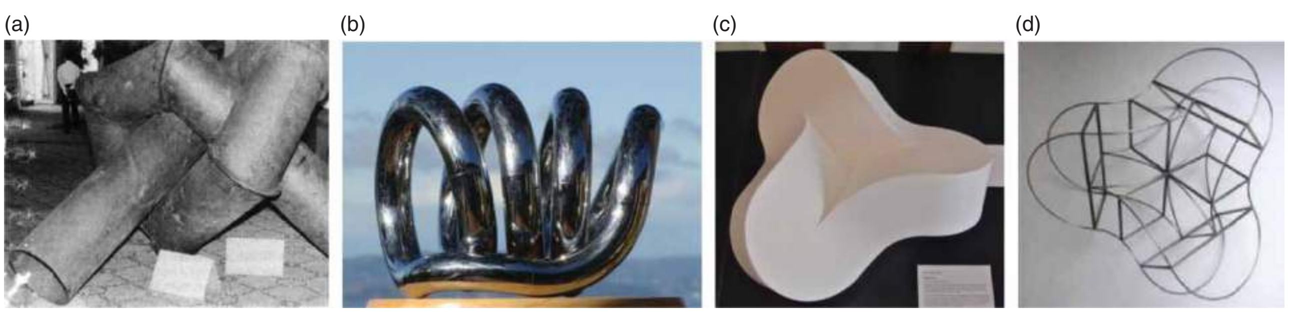

Figure 1. Sculptures built from modular tube segments: *Fit to Be Tied* by Frank Smullin (1980); (b) *Museum Tangle* by Richard Zawitz (1982); (c, d) *Borsalino* by Henk van Putten (2013).

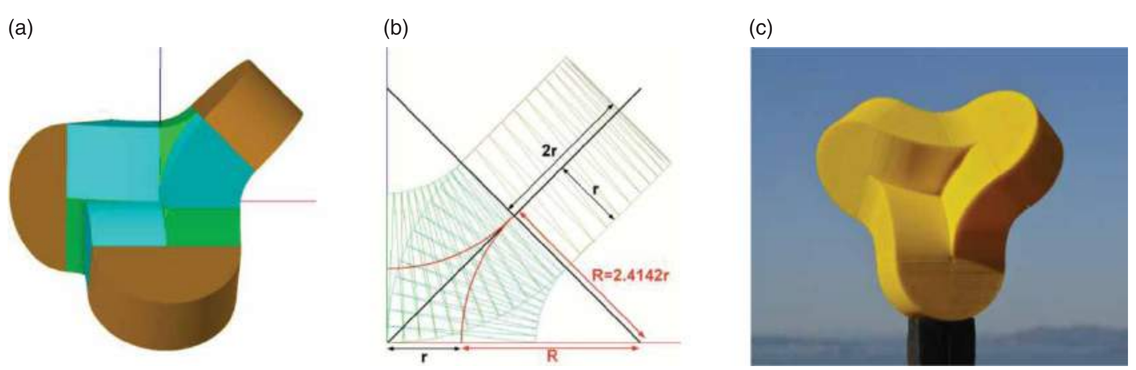

Figure 2. *Borsalino* geometry: (a) CAD model of *Borsalino*; (b) calculating the connector radius; (c) *Borsalino* assembled from nine plastic pieces.

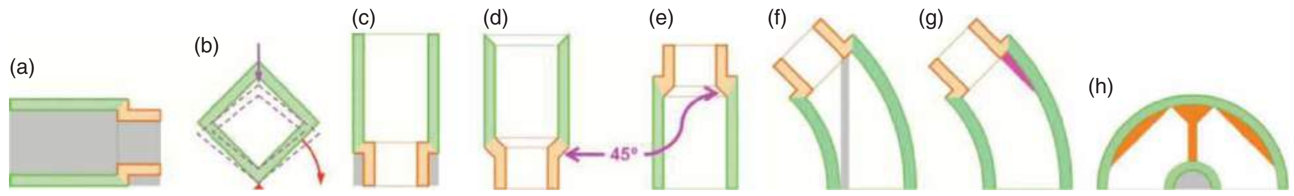

Figure 3. Fabrication options: (a) flat square tube; (b) tube on edge; (c) vertical tube, (d, e) with 45° flange transitions; (f) curved segment with sharp inner transition and (g) tapered-off; (h) end-cap.

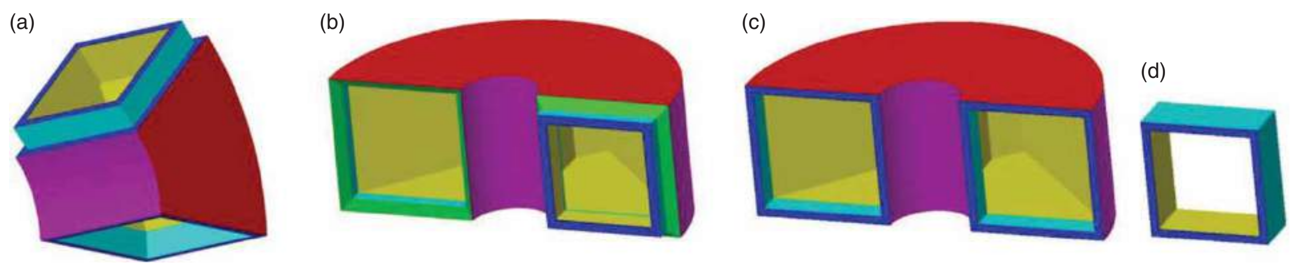

Figure 4. Fabrication options: (a) standard curved connector with built-in sleeve; (b) an end-cap with one tapered male sleeve; (c) expanded end-cap with two female ends; (d) separate, insertable sleeve.

support material at all. However, praxis shows that such a precariously balanced part will very likely get knocked over during the build process; – and even if it didn't, the resulting cross section would no longer be square due to some unavoidable sagging of such cantilevered surfaces.

It is much better to build such tubular elements in the vertical direction. A small amount of scaffolding may be required where the connection sleeve transitions into the main tube (Fig. 3c). But even this can be avoided, if at the male end the sleeve transitions from the nominal sleeve profile to the outer wall with a taper of 45° (Fig. 3d). However, such a 45° taper is not very desirable; it leaves very visible gaps where two pieces are joined. Even more advantageously, the connector parts can be built without any scaffolding, if the female end points downward (Fig. 3e); all the visible, outer tube diameter transitions can then be kept square and planar.

A vertical build orientation also works well for the curved connectors (Fig. 4a), since they don't bend through more than 45°. Again, the female end is oriented downward. An internal, downward pointing wedge near the male end might prompt the construction of a thin supporting wall below it (Fig. 3f). This can be avoided by asymmetrically tapering the offending tube diameter transition at 45° into the inner wall of the curved main tube (Fig. 3g).

In the end-caps, which turn through 180°, it is more difficult to avoid completely the use of any support material. The inner cylindrical surface could be remodeled with some kind of "cathedral ceiling" sloping at 45° (Fig. 3h); but the required central supporting wall would block the free passage through this tubular element. Also, if the end-cap uses a slightly larger turning radius, then we cannot avoid some scaffolding to support the smaller concave cylinder formed by the outer surface (Fig. 3h). Adding flanges to one or both ends of such end-caps (Fig. 4b) also increases the need for scaffolding. It then becomes advantageous to build all end-caps with two female ends (Fig. 4c) and to build separately a copious number of insertable sleeves (Fig. 4d) that can turn such an end into a male connection.

4. *Rhombic Borsalinos*

While playing with the various physical pieces, it transpired that stretching the connection in the middle of each pair of curved connectors in the regular, tight *Borsalino* loop could lead to another interesting configuration. The two side-by-side square end-cross-sections, which previously were connected by a tight end-cap, are shifted past one another until they are located corner-to-corner (Fig. 5a). Now these two ends can be closed off with a new piece, called a "rhombic end-cap", which sweeps the square cross section along a half-circle parallel to one of its face diagonals. This geometry can also be understood as sweeping a "rhombic" cross section, i.e., a square with an azimuthal rotation of 45°, along an arc with a radius enlarged by $\sqrt{2}$. Figure 5b shows the result.

These new rhombic end-caps are by themselves attractive new building modules. We can also use them as enlarged end-caps when tracing out the whole *Borsalino* shape with a consistent rhombic sweep along the composite sweep curve scaled up by $\sqrt{2}$. This requires the fabrication of six new connector pieces (Fig. 5c).

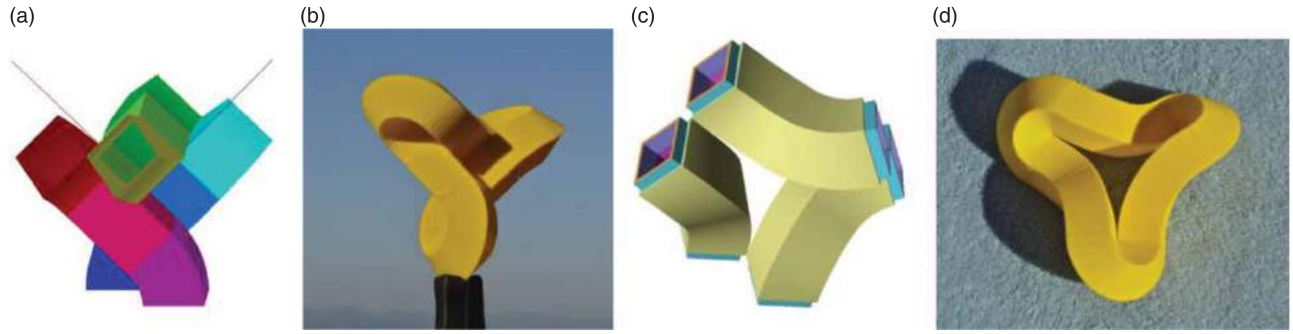

Figure 5. *Rhombic Borsalinos*: (a) an extension between a pair of connectors leads to (b) a *Flipped-over, Rhombic Borsalino Loop*; (c) diagonal bending of the connectors leads to (d) a *Rhombic Borsalino*.

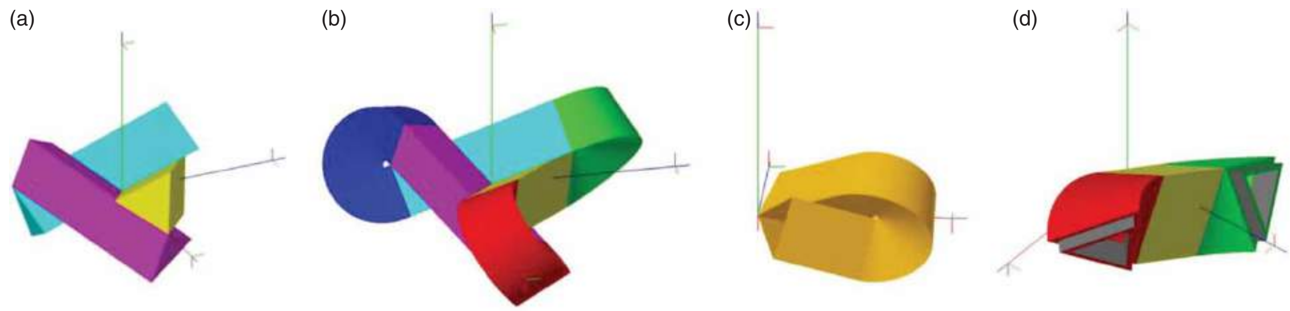

Figure 6. *Bow-Tie Loop* construction: (a) tight tangle of prismatic beams; (b) helical end-caps added; (c) a single bow-tie lobe; (d) 3-segment connector between standard 180° end-caps.

Figure 5d shows the complete, enlarged *Rhombic Borsalino*, – another instance of an attractive piece of sweep-geometry.

5. *Bow-Tie Loops*

The shape shown in Figure 5b seems to consist of three tight "Bow-Tie" lobes, where the square beam coming into the end-cap turn lies flush against the beam coming out of the turn, and the turn itself sweeps through more than 180°. Such *Bow-Tie* lobes can also be constructed using beams with different cross sections, – e.g., with the shape of an equilateral triangle. Thus one may try to connect several such *Bow-Tie* lobes snuggly into a symmetrical, twisted loop. Such geometries are best designed from the center outwards. Figure 6 illustrates the construction principle: One may start with three triangular prisms, where each has one edge lined up along the z-axis. Then each prism is rotated around its horizontal symmetry axis going through the origin until it comes into face-to-face contact with its two neighbors (Fig. 6a). The prisms are cut to the length where their outer edges intersect. Now, helical end-caps are added, so as to connect pairs of triangles that share a vertex (Fig. 6b). Figure 6c shows a single *Bow-Tie* lobe resulting from this construction. To keep the emerging set of parts as modular as possible, the overall shape can be decomposed in a different manner: The helical end-cap itself is decomposed into two small curved connector pieces attached to a standard, semi-circular end-cap sweeping through 180°; the latter is a part that can be re-used in many other configurations. Therefore these two extra connector pieces are attached to the central straight potion of the sweep to form a twisted, curved connector (Fig. 6d), which is custom made for this special *Bow-Tie Loop* with a triangular cross section.

This construction can be generalized to more than three triangular beams and to beams with other cross sections. When one starts with four triangular prisms symmetrically positioned around the z-axis, less of a rotation is required to bring all prisms into face-to-face contact with their two neighbors. Again the prisms are truncated where their outer edges intersect, and pairs of adjacent triangles are closed off with suitably twisted end-caps. The first physical *Bow-Tie Loop* with four triangular beams joining in the center (Fig. 7a) was constructed from four complete *Bow-Tie* lobes (Fig. 7b). But the skewed directions of the two prismatic connection sleeves made the assembly of the whole sculpture very difficult. Thus the approach exemplified with Figure 6d is much preferred. It was subsequently used in the construction of a 5-lobe *Bow-Tie Loop* (Fig. 7c), where five triangular prisms pass each other symmetrically around

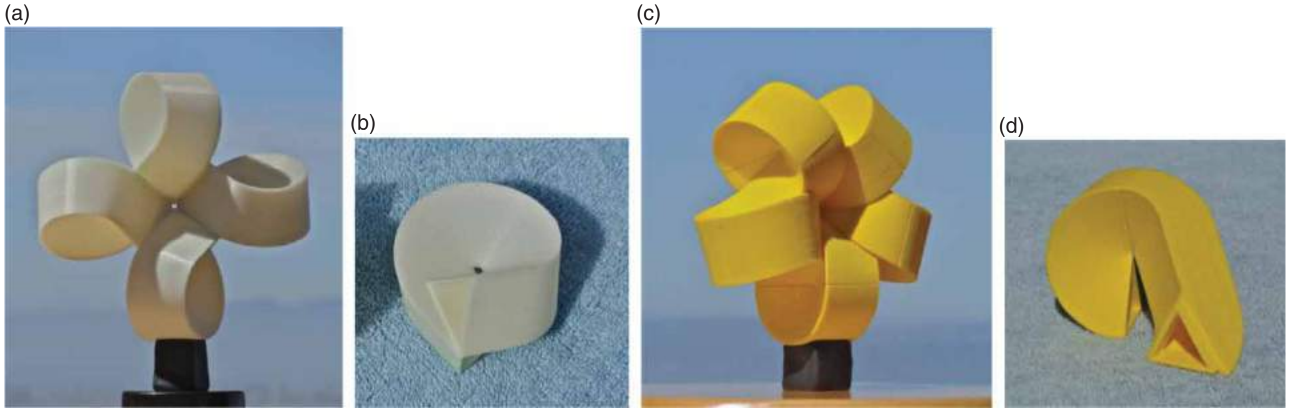

Figure 7. *Bow-Tie Loops* with three (a) and four (c) lobes and the components from which they were built (b), (d).

the center. Thus this geometry is partitioned into ten parts: five regular, re-usable end-caps that turn through 180° (Fig. 8a) and five tailor–made twisted S-shaped connectors (Fig. 7d) in the spirit of Figure 6d. This sculpture is easy to assemble. The same approach was also used to construct a 3-lobe *Bow-Tie Loop* using pentagonal prism beams.

6. Adding twisted and helical pieces

Another set of experiments, performed together with Michelle Galemmo [5], explored what shapes emerge when a triangular cross section is swept along Henk van Putten's classical *Borsalino* curve. It turns out that in this case the curved connector pieces bending through 45° also need to be given a twist of 15° to make overall smooth surface connections. For the square-sectioned *Borsalino*, a pair of curved connectors performs a topological rotation of the prism faces equivalent to a twist of 90°. For the triangular cross section, such a cyclic face re-assignment would be equivalent to a twist of 120°, and thus an actual twist of 15° has to be introduced into each connector part to correct for this fact. The end-caps (Fig. 8a), previously employed in Figures 7c and 7d, were re-used to produce the *Tria-Borsalino* loop shown in Figure 8b.

If the azimuth angle of the cross-section is changed by 180° along the whole sweep, one obtains another symmetrical geometry for the end-caps, referred to as "*Type II*." The modified end-cap and the resulting *Tria-Borsalino* are shown in Figures 8c and 8d. Note that the required connectors are different for the two types of *Tria-Borsalinos*, since the starting azimuths of the sweeps differ by 180°.

Once the notion of twist had been introduced, it seemed natural to explore what this might bring to the original *Borsalino* shape with a square cross section. Even though Figures 5b and 5d may have a somewhat twisted look, these generalized cylinders are still minimum-torsion sweeps. However, the *Rhombic Borsalino* (Fig. 5d) is loose enough, so that there is room to add actual twist into the six curved connector pieces. To keep the connections to the rhombic end-caps in the same orientation, we must give each connector pair a total twist of 90°. Because of the asymmetry introduced by the male and female coupling sleeves, this leads to two new connector parts (Fig. 9a,b). With such a pair we can form twisted connections between consecutive end-caps. Figure 9c has a single twisted link at the bottom; Figure 9d has all three links twisted.

Modeling these twisted connectors offers some challenges. The outside surface should be nice and continuous

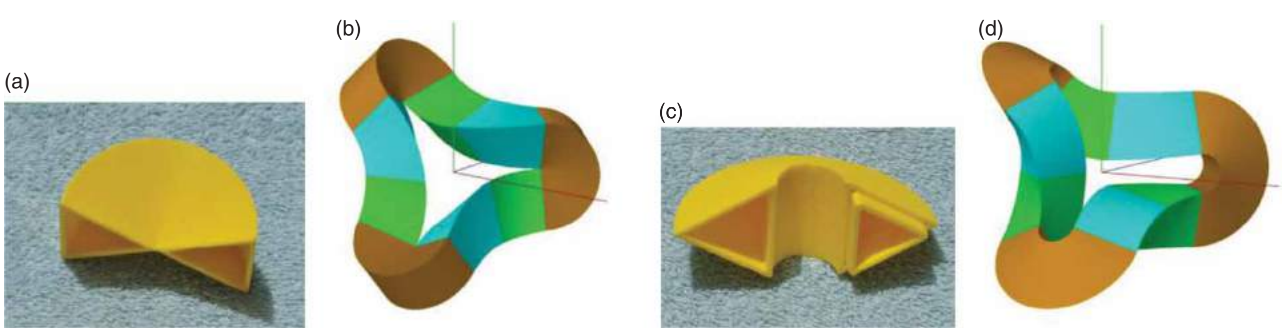

Figure 8. *Tria-Borsalinos*: (a, b) *Type I* end-cap and assembly; (c, d) *Type II* end-cap and assembly.

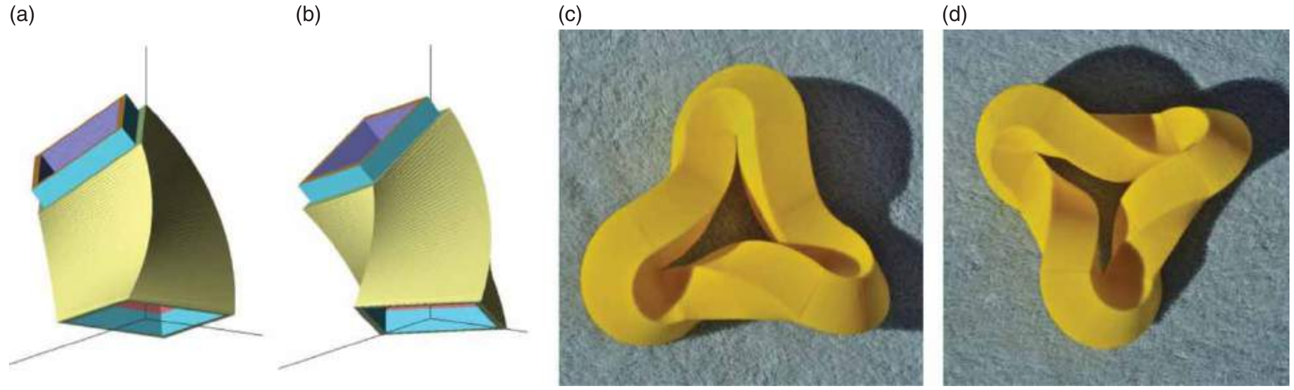

Figure 9. *Twisted Rhombic Borsalinos*: (a, b) the twisted connector components; and the results: with one twisted branch (c) and with three twisted branches (d).

when these parts are chained together; thus it wants to be part of a continuously twisting helical structure. On the inside, however, there should be short, straight sleeve sections at both ends, so that these parts can fit together with any of the other "LEGO®-Knot" parts. We found that a good way to model the inner surface is with a sweep along a cubic Bézier curve, where the end-points, the end-tangents, and the azimuthal orientation are carefully adjusted to match up seamlessly against the straight sleeve sections.

7. Sculpture emulations

On Henk van Putten's Facebook homepage [17] more sculptures can be found that are composed mostly of the same geometrical elements described earlier in this article. Figure 10 shows how well these modules can approximate three more of Henk van Putten's sculptures. The first two examples (Fig. 10a,b) employ only the two modules used in the original *Borsalino*. The right-most example (Fig. 10c) also uses the expanded curved connector piece used for the loose *Borsalino* (Fig. 4c). To obtain better

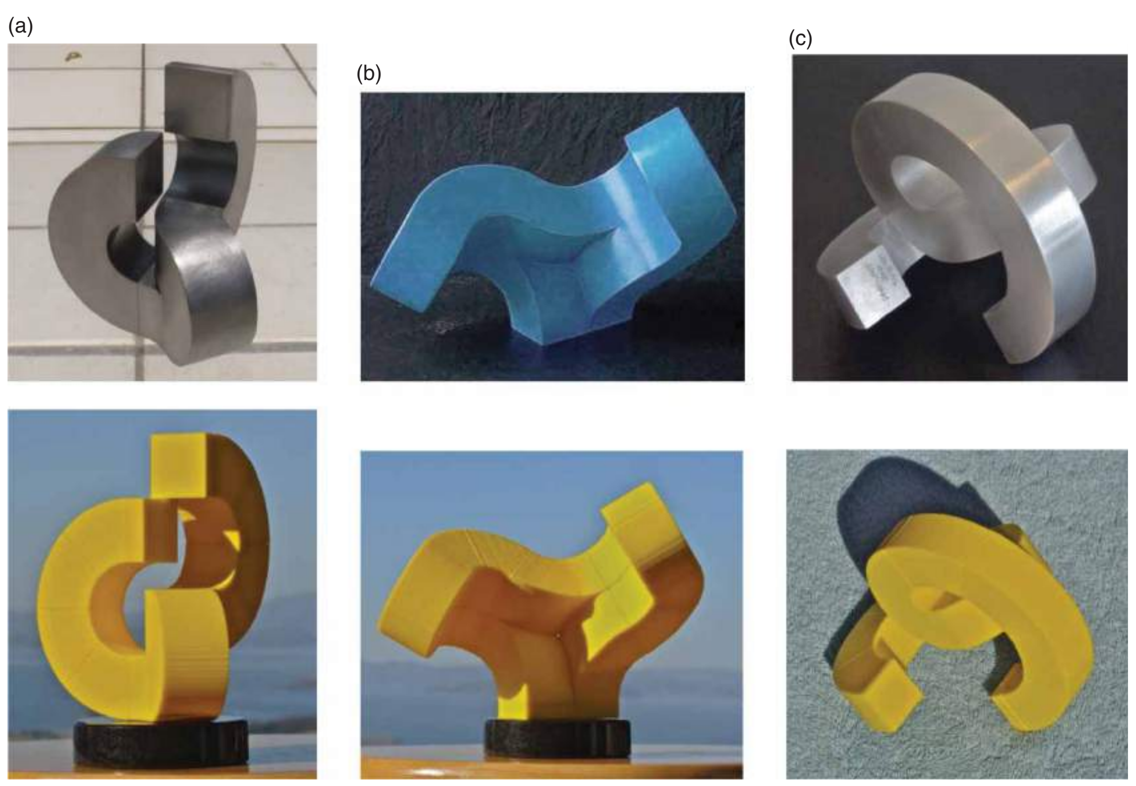

Figure 10. Inspirational sculptures by Henk van Putten found on his Facebook timeline [17] (top row) and the emulation of these sculptures with "LEGO®-Knot" pieces (bottom row).

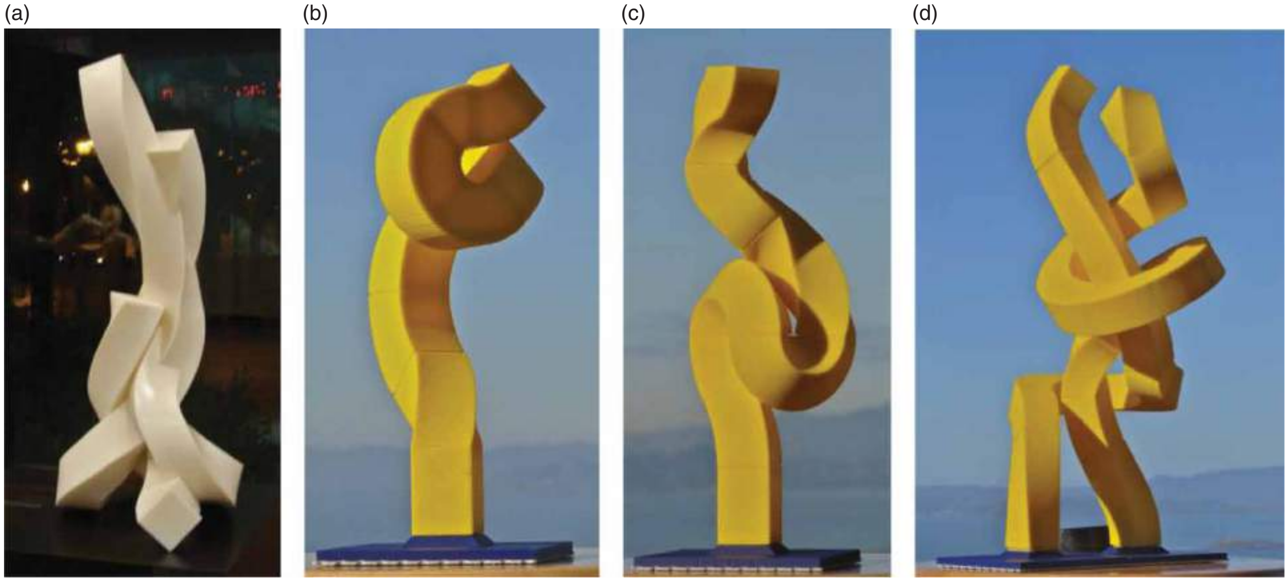

Figure 11. (a) Sculpture by Beasley [3] inspiring free-standing "LEGO®-Knot" constructions (b–d).

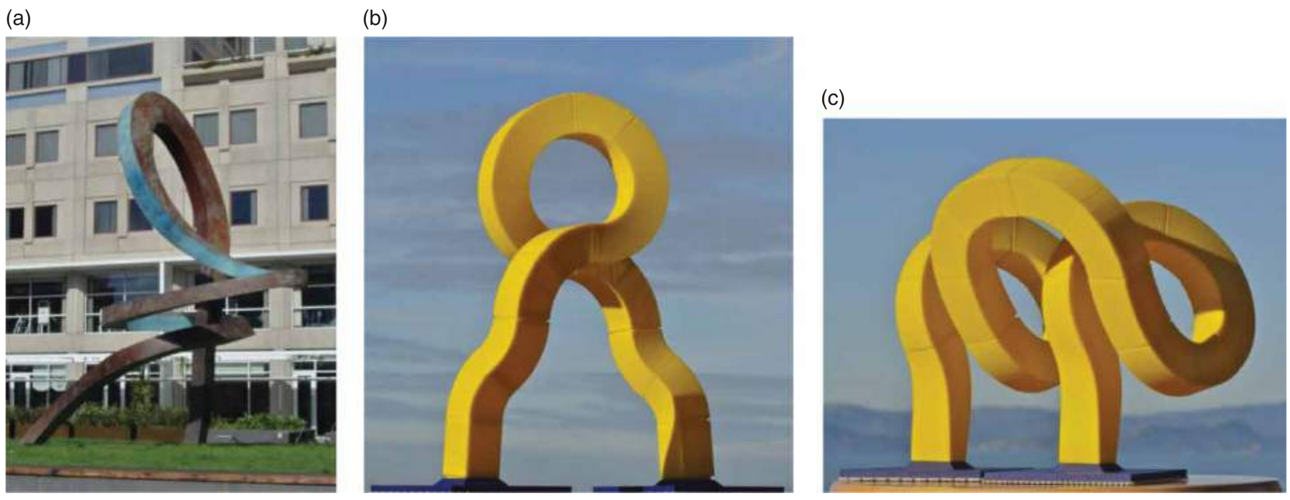

Figure 12. (a) Sculpture by Krawczyk [6] inspiring free-standing LEGO®-Knot constructions (b,c).

visual agreement, a 1″-square end-cover was fabricated to close off the hollow tube-ends.

The enlarged set of parts resulting from making various derivatives of the original *Borsalino* geometry, yields enough flexibility to emulate also various tubular sculptures by other artists, such as Bruce Beasley [2], Jon Krawczyk [6], or Paul Bloch [4].

In October 2013 Bruce Beasley opened *Coriolis* [3], a *3D-Printed Art Exhibition* at the Autodesk Gallery in San Francisco. All exhibited sculptures were basically sweeps of a square cross-section along one or more intricate free-form space curves. While it is not possible to model the continuously varying curvature exhibited in most of these sculptures with our modular "LEGO®-Knot" parts, they can still serve as inspiration. Figure 11a shows a vertically thrusting sculpture by Beasley and a couple of "LEGO®-Knot" constructions inspired by it. For this kind of sculpture a special (blue) platform was fabricated, which holds the lowest part in an upright position (Fig. 11b-d).

Also in downtown San Francisco, one can admire three sculptures by Jon Krawczyk [6]; these are also progressive sweeps with a square profile (Fig. 12a). Since they join the ground with two legs, a second (blue) platform was fabricated, and a variety of free-form sweeps flowing from one to the other were assembled (Fig 12b,c).

Several of Paul Bloch's sculptures [4] are dominated by helical elements. A modular system cannot reproduce the continuously changing curvatures found in Bloch's work (Fig. 14a); all helices need to be regular and of the same

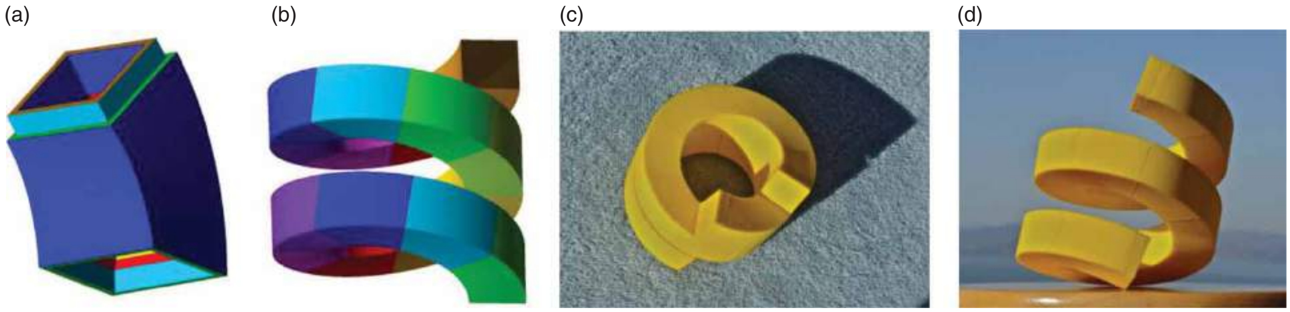

Figure 13. Helices: (a) one helical piece; (b) two turns of a helical spiral formed with 16 pieces; (c) interlaced helical sweeps; (d) serially connected, left-turning spiral loops.

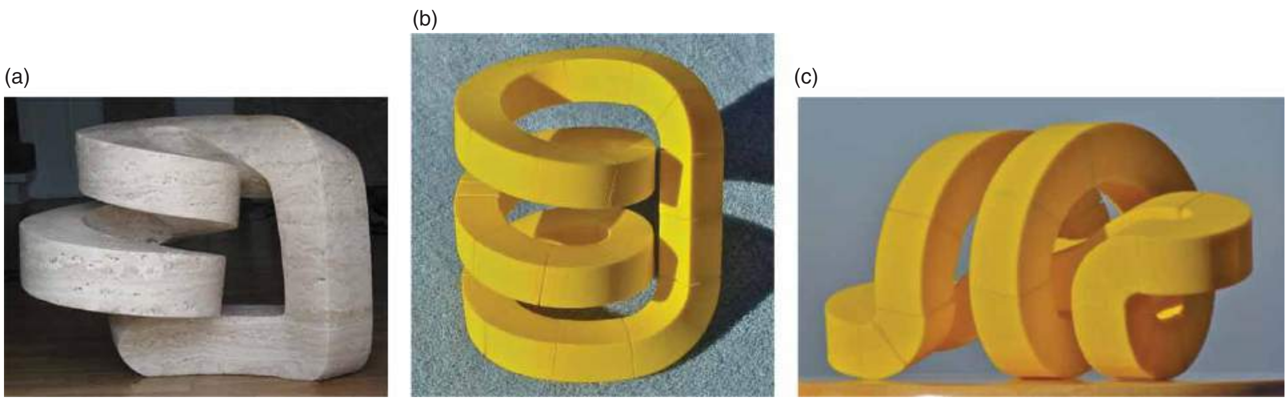

Figure 14. (a) *After Wright* by Bloch [4]; (b) an emulation thereof; (c) another helical closed loop.

type (Fig. 13b). Thus a "general-purpose" helical component was introduced (Fig. 13a). This component sweeps through 1/8 of helical turn (Fig. 13b), and the pitch of this helix was set so that two identical helices could be tightly intertwined (Fig. 13c). So far, only left-handed spirals have been fabricated (Fig. 13d).

This helical module now enables an approximation of Bloch's *After Wright* sculpture (Fig. 14a). Serendipitously, smooth closure could be achieved for a 2.5-turn helical spiral by adding two straight pieces, two standard curved connectors, and two rhombic curved connectors (Fig. 14b). There is also a way to close off a 2-turn helix with a path that goes through the center of this corkscrew. However, there is some strain in this assembly, as revealed by the gaps near the sharp bends (Fig. 14c). Creating a smoothly closed sweep through 3D space with our limited set of different tubular modules is a nontrivial challenge: Six degrees of freedom (x, y, z, and 3 angles) have to be matched to obtain smooth closure. This challenge becomes more severe for tightly wound, knotted configurations.

8. Non-trivial knots

All of the shapes presented so far have been open-ended sweeps or simple loops equivalent to the un-knot. This section discusses the difficulties of using a small, "generic" set of building blocks to construct compact, well-formed, symmetrical models of mathematical knots.

The simplest true knot in the Table of Mathematical Knots [8] is the trefoil knot (Knot 3_1). To form a nice, tightly wound realization of this knot, a good start is to use six of the above helical pieces to form ¾ turns of a helical spiral. Three such helical arcs can cover about 85% of the envisioned trefoil sweep. They are placed into a D3-symmetric [10] configuration, and the rotation around the three C2-axes (passing between the lime and green colored pieces in Figure 15a) as well as the distances of the three arcs from the origin are adjusted interactively. The goal is to line up, as best possible, the three pairs of arc-ends that need to be connected. However, the small set tubular modules at hand were insufficient to construct a graceful closure between the three helical arcs. Exploiting the rapid turn-around provided by layered manufacturing, a new custom-designed part was introduced that fit nicely in between the three helical arcs (shown in magenta in Fig. 15a). Figure 15b shows the full physical realization of a modular trefoil knot.

The second entry in the Knot Table [8] is the figure-8 knot (Knot 4_1). The most symmetrical configuration of this knot (Fig. 15c) has 4-fold rotational glide symmetry around the z-axis (S4-symmetry [10]). The helical

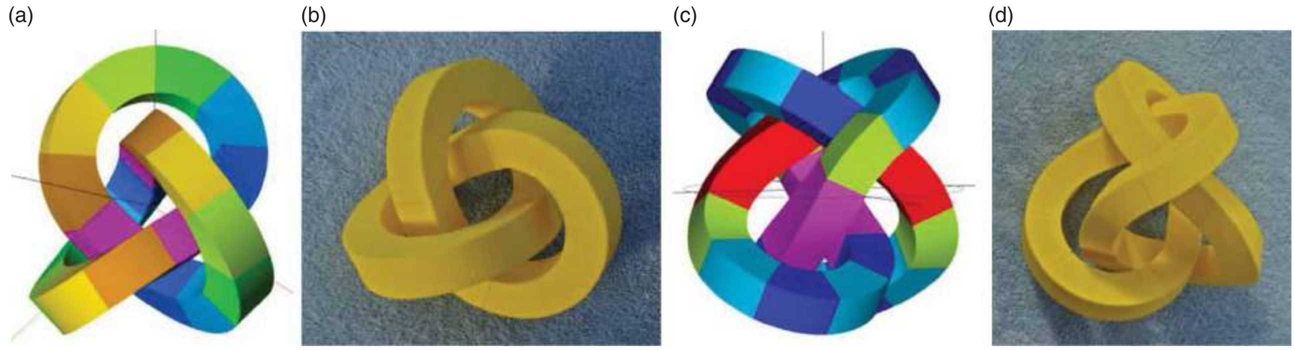

Figure 15. Non-trivial knots: (a,b) trefoil knot (Knot 3_1); (c,d) figure-8 knot (Knot 4_1).

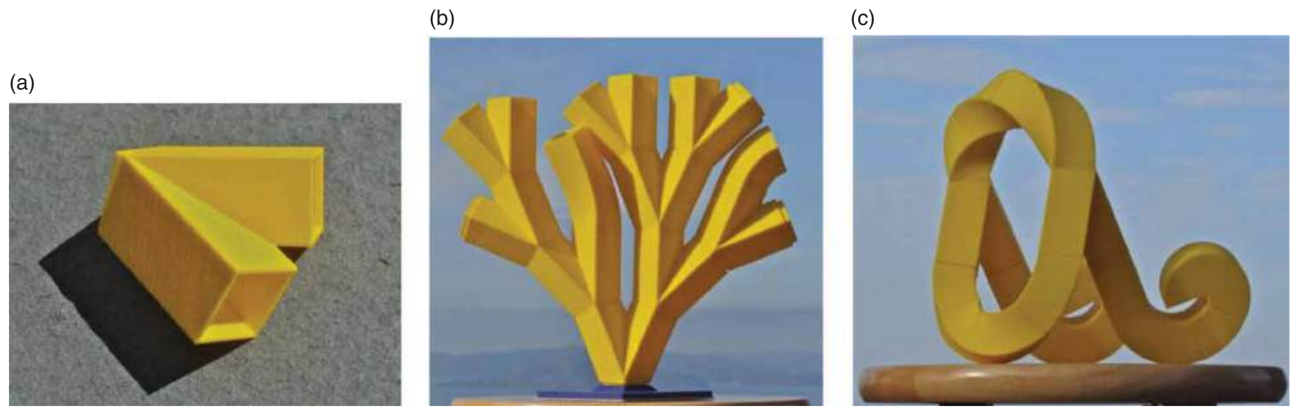

Figure 16. A new branch part (a) allows to make (b) tree structures and (c) general graphs.

arcs employed in the trefoil knot, are of no use here: The figure-8 knot is non-chiral, i.e., it is its own mirror image, and at this time only left-handed helices had been fabricated. Instead, four planar, hemi-circular loops are constructed from four curved connector pieces (blue and cyan in Fig. 15c). To bend the ends of these planar arcs more closely into the direction in which they need to join up with a corresponding end, a rhombic connector piece (green) has been added at one end. Again a new custom piece with the appropriate amount of bending and twisting is needed to obtain graceful closure. However, because of the amphichiral nature of this knot, two pairs of mirror images had to be fabricated – shown in magenta and red in Figure 15c. This then leads to a rather smoothly curved knot construction (Fig. 15d).

9. Branching out

At some point the lineal nature of all these assemblies started to feel too confining. The branch component shown in Figure 16a was introduced, and it enabled the construction of "tree-" or "coral-like" structures (Fig. 16b) as well as arbitrary graphs, such as *Girl with Curls* shown in Figure 16c.

To make sculptural realizations of regular cubic graphs (with all valence-3 vertices), such as the edge graphs of some of the Platonic solids, using only the available "general-purpose" parts, is even more challenging than making well-formed knots. Now there are many more branches that need to be closed with a good match of all six degrees of freedom (x, y, z, and 3 angles). Figure 17a shows a moderately successful construction equivalent to the simple tetrahedral edge graph. It resulted in a rather loopy structure, which however displays 4-fold D2-symmetry. By introducing again one custom-designed part, a more compact and streamlined representation of this graph was obtained, in which all branches close smoothly (Fig. 17b). Figure 17c shows a reasonably compact realization of the edge graph of a cube; but it has much less symmetry than the 48-fold symmetry of a plain cube. The newly introduced branch module, with an angle of 45° between its legs (Fig. 16a), is definitely not an optimal component for the construction of the edge-graphs of the regular polyhedra.

10. The fit to LEGO®-DUPLO

By pure serendipity, the chosen sleeve dimension of 1 square inch just fit around four nibs of the LEGO®

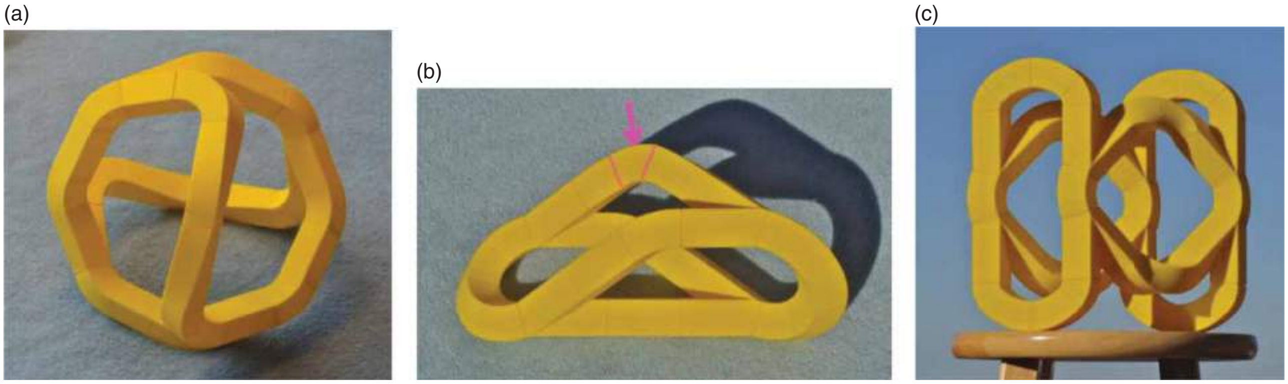

Figure 17. Geometrical graphs equivalent to the tetrahedral (a, b) and the cubical (c) edge graphs.

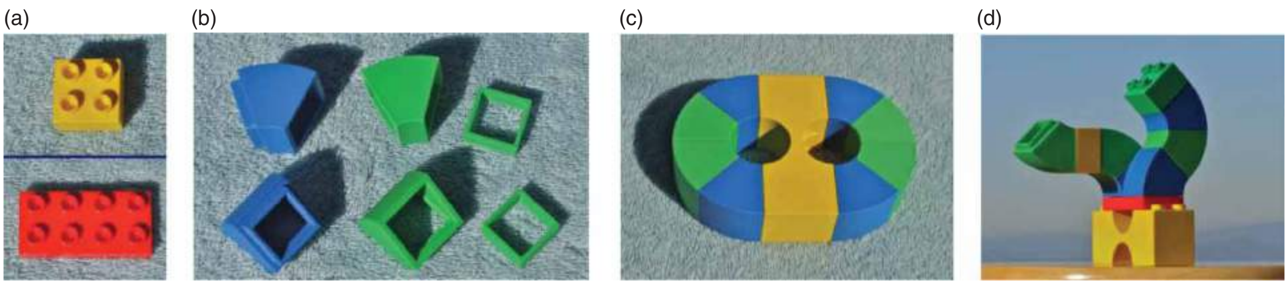

Figure 18. (a) LEGO® DUPLO pieces; (b) matching curved connectors; (c, d) resulting assemblies.

DUPLO system (Fig. 18a). However, to obtain a smooth match of the outer walls with the LEGO® DUPLO parts, which are based on a 32 mm grid [7], the wall thickness of the tubular parts has to be increased from 1.75 mm to 3.1 mm. A better design is to stick with a wall thickness of 1.75 mm and add a thickened rim at the female end to yield the needed 1″-square connection around the 4 nibs. The DUPLO stud height (5 mm) determines the lengths of the connecting sleeves. Figures 18b–d show a batch of curved connector parts that bend through 45° and mesh nicely with the LEGO® DUPLO parts.

By combining the new curved pieces with standard LEGO® DUPLO pieces, it is possible to make nice models of mathematical linkages: in particular, the Borromean rings (Fig. 19a) and the Hopf link (Fig. 19b), which are entries L2a1 and L6a4 in the Thistlethwaite Link Table [15].

When fitting some new "LEGO®-Knot" parts to the DUPLO system, the question arises whether one should use open-ended tubular modules or adopt the closed-face LEGO® approach with the 4 nibs protruding from the surface. Open-ended tubular modules are much

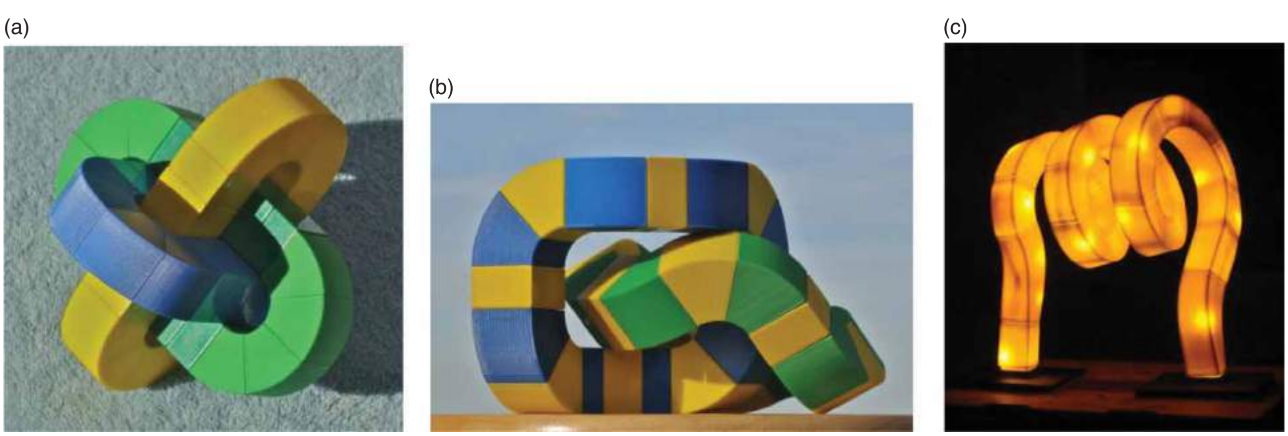

Figure 19. Linkages with DUPLO pieces: (a) Borromean rings (Link $6^3{}_2$) [9] and (b) Hopf link (Link $2^2{}_1$) (c) Glow-in-the-dark sculpture making use of the open-ended hollow tubes.

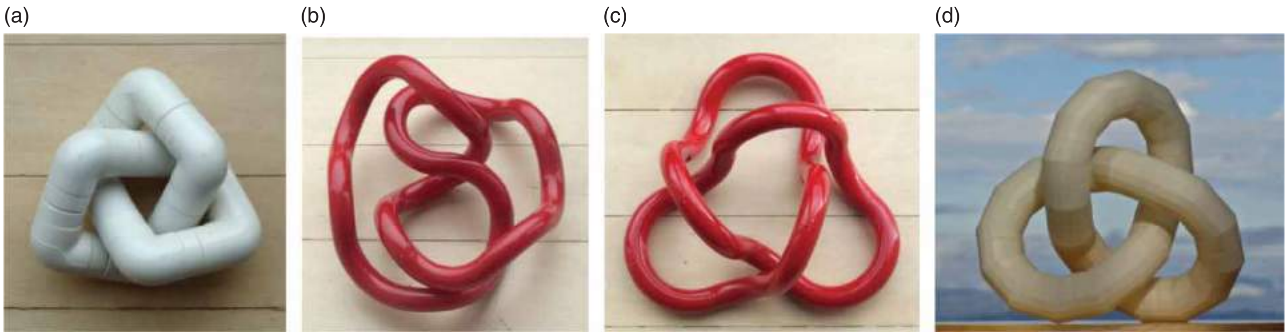

Figure 20. Trefoil knots realized with: (a) ¾-inch PVC-pipe elements, (b,c) Zawitz's *Tangle* [19], (d) 33 modules of the type shown in Figure 21a.

more amenable to fabricating these parts with a minimal amount of build material and without the use of any support material. The open tube elements also have the additional benefit that Christmas lights can be strung through them, so as to produce attractive, glow-in-the–dark sculptures (Fig. 19c).

11. Totally modular knots and links

The mathematical knot sculptures described above have been realized mostly from general-purpose "LEGO®-Knot" parts, with one or two custom-designed modules added to obtain maximal symmetry and graceful closure of a knot curve. Suppose we wanted to build several different knot models out of just one single, "universal" building block. What should this module look like?

Knot models can be built from plastic pipe elements available in any hardware store. A trefoil knot can be built from nine right-angle pipe elbows connected with nine straight pipe segments – which, however, cannot be all of the same length (Fig. 20a). Alternately, Zawitz's *Tangle* [19], which is sold as an un-knotted loop of 18 quarter-turn toroidal elements, can be broken open and reconfigured into a trefoil knot. But manipulating a closed knotted loop is rather awkward, and it is difficult to obtain symmetrical shapes; solutions with approximate C2- and C3-symmetry are shown in Figures 20b,c.

Both of these types of constructions exist in a smooth deformation space, since the circular cross-sections of the tube elements allows arbitrary torsional twisting between subsequent modules. Thus it is very difficult to know when one has achieved a knot configuration with perfect symmetry. To create a discrete solution space with a finite number of possible configurations, the module must permit only a finite number of azimuthal angles by which subsequent elements can be joined. A promising compromise, yielding distinctly discrete azimuthal angles even when fabricated on a low-end FDM machine, is based on a regular 16-gon; it still offers sufficient azimuthal options that can lead to nice and compact realizations for simple knots.

A second trade-off concerns the bending angle of such a universal tubular module. Maximal control over the shapes of the lobes that can be formed would result from a thin, wedge-like sliver; but this would then require a large number of modules for even the simplest knots. A large bending angle may result in knots built from fewer parts, but will constrain the possible geometries more severely. To find a practical compromise solution, an interactive CAD program has been developed that can chain several individual modules into larger compounds by simply specifying the azimuthal angles at subsequent module joints. Copies of these compounds can then be placed with the desired symmetry; e.g., for the trefoil knot six copies can be positioned with D3-symmetry. Then the 5 or 6 defining azimuth angles in one compound are adjusted interactively to explore whether one of the possible angle combinations can close the loop formed by the six compounds to within a small fraction of the tube diameter and with a tangent alignment of a few degrees. Extensive studies showed that a bending angle of 30° seemed to work well for the first three knots in the Knot Table [9].

Rapid prototyping was the final step in evaluating whether the knots composed with the proposed module could indeed be realized, i.e., whether the designed tubular assemblies were sufficiently flexible for the loops to close within the tolerances and rigidity of these parts and their snap-together connections. Indeed, the 16-gonal module bending through 30° (Fig. 21a) permitted the construction of the first few knots in the table with high symmetry: A D3-symmetric Knot 3_1 was constructed from 33 "universal" modules (Fig. 20d); this should be compared with Figures 15a,b. Another trefoil knot with D2-symmetry was built from 38 modules. Knot 4_1 with S4-symmetry (as in Fig. 15c,d) required 40 modules. 50 building blocks can make up Knot 5_1 with D5-symmetry (Fig. 21b). Also, a compact model of the

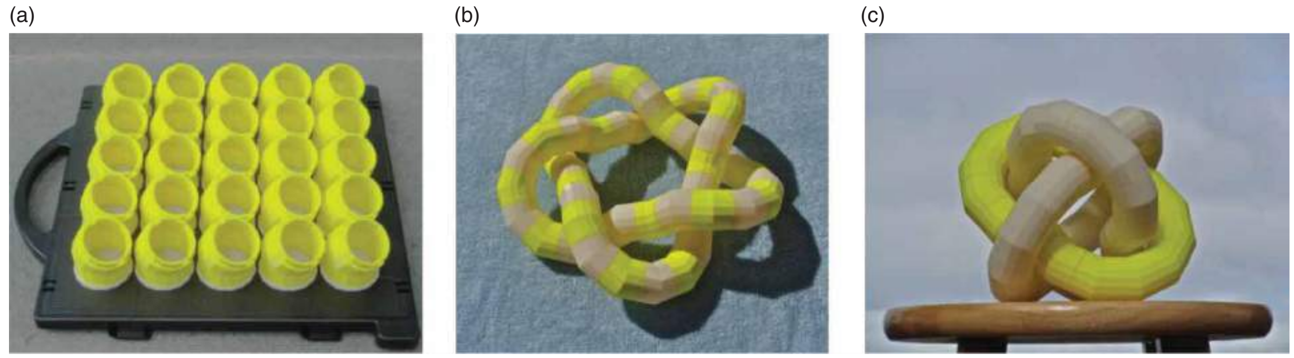

Figure 21. A single modular component (a) to construct highly symmetrical knots (b) and links (c).

Borromean link (Fig. 19a) can be assembled from 3×16 modules (Fig. 21c).

Clearly, once a useful "universal" module has been defined, this part should be mass-fabricated more cost-effectively with an injection-molding process. Then, with an ample supply of such modules in hand, the remaining open challenges revert back to the design aspect: How can one realize any envisioned knot or link with a minimal number of parts and with as nice and uncontorted a look as possible? For small knots where only 5–8 unique azimuth angles have to be set, an exhaustive search may be a viable option given today's computer power. Even an interactive search within the described CAD program was quite practical. The solutions mentioned above were all found within less than an hour of virtual experimenting, once the author had developed some intuitive understanding how a chain of modules might react to a chosen azimuth change. The virtual exploration was certainly faster than trying to find a symmetrical knot configuration by assembling the physical modules.

For more complicated knots with less symmetry, a much larger number of individual azimuth angles have to be set, and the number of angle combinations will run into the trillions. No program has yet been developed to find automatically the most compact closed-loop realization of a particular knot. The nature of this discrete solution space will require a probabilistic approach to find an acceptably good solution. Moreover, even programming an efficient search strategy based on simulated annealing will require some insightful definition of some "meta-moves;" these may be pairs or triplets of synchronized angle changes on adjacent joints that produce less "violent" motions of the end of a long chain of modules than what changing a single azimuthal angle typically will produce. There is one mitigating effect: As larger numbers of modules are strung together, the end-to-end flexibility of the assembly increases, and the required geometrical match-up needs to be less precise.

12. Summary and conclusions

In many domains of design and engineering, rapid prototyping has become an important and unavoidable step in the design process. This step is equally important for artists who create geometrical sculptures. During the last two decades the author has designed dozens of sculptures using a variety of CAD tools and then implemented many of them as small sculptural maquettes on various rapid prototyping machines. Experience has shown that, no matter how carefully the "final" designs were inspected on the computer screen, once a physical prototype became available, ways to improve the sculptures could almost always be found.

"LEGO®-Knots" are an experimental, hands-on approach for constructing a special class of tubular assemblies. A limited set of part types, all based on sweeps of a fixed cross-section along a circular or helical arc, allow the user to construct a wide variety of sculptural forms. This study started out with the two pieces required to re-create Henk van Putten's *Borsalino* shape [17]. But as soon as these first parts were in the author's hands, he wanted more of them and wanted to put them together in different ways. Shortly thereafter he also wanted to make modified *Borsalino* shapes, with rhombic cross sections and with twisted legs. The availability of a rapid prototyping machine on which such extensions could be realized within 24 hours, made this a very exciting and productive activity. New parts almost immediately inspired additional geometrical visions, and the occasional need for a special custom-designed part to complete a particular project could also be fulfilled with fast turnaround times.

A few months into the project, a slightly modified set of parts got matched to the LEGO® DUPLO system. With this integration, immediately a much richer set of shapes could be constructed. This points the way to the most effective use of 3D printing: If at all possible one should try to make use of already existing building blocks and just focus on designing and fabricating the critical parts that the existing system cannot deliver.

Large savings in cost and turn-around time can result, as so beautifully demonstrated by the "faBrickator" system [8]. Even for the explorations carried out in the author's "LEGO®-Knot system," whether it was the emulation of some sculpture of a famous artist or the smooth closure of a knotted sweep through 3D space, there were often one or two extra components that needed to be introduced in order to complete a particular task. Without access to a layered manufacturing machine or to a 3D-printing service, this could result in frustrating delays. Obtaining the needed parts within a day or two, keeps the excitement alive, and it often stimulates new ideas for what should be tried next. Thanks to the new technologies of additive machining, many more people can now experience this exciting "design and build" mode, which seems to amplify one's creativity.

ORCID

Carlo H. Séquin http://orcid.org/[0000-0002-0060-2162]

References

[1] Afinia, H479 3D Printer, – http://www.saelig.com/product/RO023.htm?gclid=CJ75oJTQ27sCFZKGfgod50gASw
[2] Beasley, B.: Sculptures 1960-Present, – http://www.brucebeasley.com/home.htm
[3] Beasley, B.: Coriolis, a 3D-Printed Art Exhibition, – http://www.businesswire.com/multimedia/home/20131030006167/en/#.UsYOkFDAYgE
[4] Bloch, P.: Portfolios 1,2,3, – http://www.paulbloch.com/
[5] Galemmo, M. and Séquin, C. H.: Tria Tubes. Bridges Conf. Proc, Seoul, Korea, Aug. 14–19, 2014, pp 373–376, – http://archive.bridgesmathart.org/2014/bridges2014-373.html
[6] Krawczyk, J.: Sculpture; Public Works, – http://www.krawczyk-sculpture.com/fly.aspx?layout=gallery#
[7] LEGO® Answers, Exact dimensions of a DUPLO brick, – http://bricks.stackexchange.com/questions/1588/what-are-the-exact-dimensions-of-a-duplo-brick
[8] Mueller, S. et al.: faBrickation: Fast 3D Printing of Functional Objects. CHI 2014, Toronto, Canada – http://www.stefaniemueller.org//wp-content/themes/f8-lite/images/faBrickation/chi-2014-mueller-faBrickation.pdf
[9] Rolfsen Knot Table, – http://katlas.math.toronto.edu/wiki/The_Rolfsen_Knot_Table
[10] Schönflies notation for finite point groups, – http://en.wikipedia.org/wiki/Crystallographic_point_group
[11] Séquin, C. H.: Tubular Sculptures. Bridges Conf. Proc, Banff, Canada, July 25–30, 2009, pp 87–96, – http://archive.bridgesmathart.org/2009/bridges2009-87.html
[12] Séquin, C. H. and Galemmo, M.: LEGO-Knots. Bridges Conf. Proc, Seoul, Korea, Aug. 14–19, 2014, pp 261–270, – http://archive.bridgesmathart.org/2014/bridges2014-261.html
[13] Shapeways: 3D printing service, – http://www.shapeways.com/
[14] Smullin, F.: Fit to Be Tied. Reconfigurable sculpture, – http://collections.si.edu/search/results.htm?q=record_ID:siris_ari_319459
[15] Thistlethwaite Link Table, – http://katlas.math.toronto.edu/wiki/The_Thistlethwaite_Link_Table
[16] van Putten, H.: Limitation or addition? Experience-centered Approach and Visuality in The Education of Mathematics and Physics, Kaposvár University, 2012, pp 193–194.
[17] van Putten, H.: Facebook home page, – https://www.facebook.com/agnusandhenk
[18] Venus of Willendorf, – http://en.wikipedia.org/wiki/Venus_of_Willendorf
[19] Zawitz, R.: Museum Tangle, – http://www.richardxzawitz.com/