Adaptive Tetrahedral Mesh Generation of 3D Heterogeneous Objects

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ABSTRACT

In this paper, a step-wise tetrahedral mesh generation method is introduced to solve the adaptive meshing problems of 3D heterogeneous objects, in which both the material heterogeneity and the geometric complexity have to be taken into account. In light of that, the proposed meshing strategy embraces three steps: initial mesh generation, material-oriented refinement, and geometry-oriented refinement. In the material-oriented refinement, a global refinement algorithm based on optimal Delaunay triangulation (ODT) is designed to control the mesh adaptation in terms of the material distribution. Compared to the refinement strategies based on local modifications, the global refinement algorithm takes full advantages of the material heterogeneity information and distributes the material composition variation over elements as equally as possible. As a result, the number of elements is reduced to a great extent regarding a given material threshold. Implementations show that the proposed approach guarantees desirable mesh adaptation as well as high mesh quality.

Keywords: adaptive meshing, 3D heterogeneous objects, global refinement, ODT

1. INTRODUCTION

In recent years, heterogeneous materials have attracted a lot of attention for their superior properties over homogeneous materials [14,17,19]. To design, analyze, and optimize the behavior of heterogeneous materials, the finite element method (FEM) as an effective numerical method has been immensely utilized [8,13,16]. Although tremendous efforts are devoted to finite element analysis (FEA) of heterogeneous materials in the past two decades, little attention has been focused on mesh generation, which is an essential part of the FEA procedure. For the sake of simplicity, classic mesh generation methods targeted on homogeneous objects are often directly applied into the domain of heterogeneous models. The meshes generated by these traditional methods, however, either result in poor simulation accuracies (as they fail to characterize the material heterogeneities) [2], or introduce denser elements than desired, significantly degrading the computational efficiency [12]. To solve such problems, specific mesh generation methods for efficient and robust FEA of heterogeneous materials are called for.

Zhang et al. [25] proposed an automatic 3D meshing strategy for heterogeneous objects. An octree-based isocontouring method is utilized to construct adaptive and high-quality tetrahedral/hexahedral meshes that conform to the material boundaries. Unfortunately, only multiple material objects (see Fig. 1(a)), which are very primitive in terms of material heterogeneities, are taken into account. Cuillière et al. [10] developed an automatic approach based on the advancing front method to generate unstructured tetrahedral meshes in the context of composite or heterogeneous geometry. The obtained meshes, however, are only suitable for FEA of multiple materials.

Functionally graded materials (FGMs) [18], whose material heterogeneities vary gradually within the domain of interest as shown in Fig. 1(b), usually outperform multiple materials. Therefore, meshing strategies that concentrate on FGM models warrant further exploration. To generate efficient meshes on FGM models, adaptive meshing methods are often used. Shin [23] proposed an iso-material method to tackle the adaptive meshing problem of FGM models. In his work, iso-material contours of an FGM model are first created, converting continuous material distribution into step-wise variation; triangular mesh is then constructed inside each homogeneous

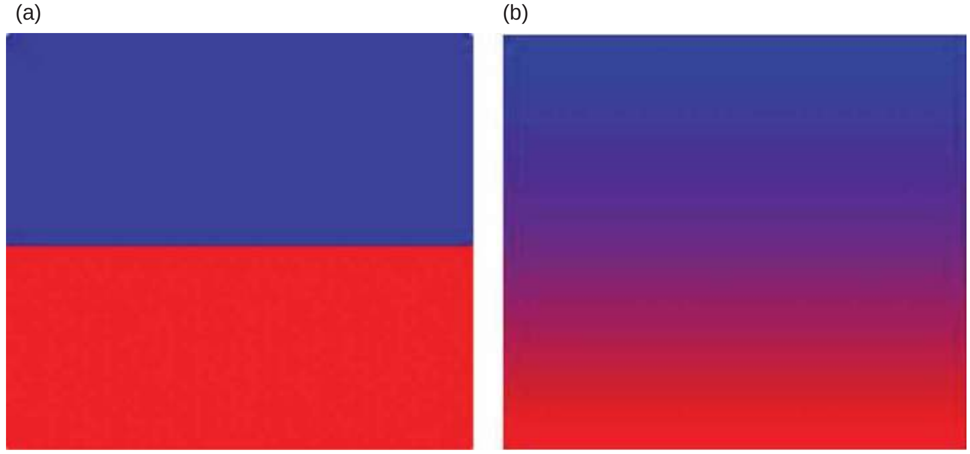

Fig. 1: Heterogeneous material distribution examples: (a) Multiple material model; (b) FGM model.

region formed by adjacent iso-material contours. More recently, Chiu et al. [9] introduced an adaptive meshing method based on quadtree for complex FGM models. In their work, they proposed to use a material threshold to evaluate if a quadrant is homogenous or quasi-homogenous. The subdivision of the domain was recursively processed until all the quadrants satisfy the material threshold as well as the geometric resolution. The obtained quadrants are then converted to triangular mesh by Delaunay based methods. One common disadvantage of these two studies is that the authors have paid duly attention to 2D FGM objects, yet having not touched on 3D FGM objects that are more general and complex in practice.

To the best of our knowledge, there exist few systematic meshing strategies that focus on the mesh problems of 3D FGM models although some researchers claim that their 2D adaptive meshing method can be easily extended to 3D cases. We argue that this kind of claim is not necessarily valid. First, it is significantly more difficult to generate high-quality 3D meshes (tetrahedral meshes) than its 2D counterpart (triangular meshes). For instance, the regular tetrahedron does not tile 3D volume, while the equilateral triangle tiles the 2D plane; unlike the triangular mesh, even well-spaced vertices can create degenerate tetrahedra (e.g. slivers) in the tetrahedral mesh. Second, the consumption of computational resource on mesh generation of 3D FGM models is excessively larger than the 2D cases. As the dimension of FGM models changes from 2D to 3D, the growth of the computational cost (including CPU time and memory) is barely linear but often involves steep nonlinear increases. As a result, effective mesh generation methods for 3D FGM models call for further examination.

In this paper, we propose an adaptive tetrahedral mesh generation method for general 3D heterogeneous models. In order to tackle the material heterogeneity and the geometric complexity respectively, we separate the proposed meshing strategy into three steps, which are initial mesh generation, material-oriented refinement, and geometry-oriented refinement. In the material-oriented refinement, a global mesh adaptation algorithm based on optimal Delaunay triangulation (ODT) [7] is developed to control the mesh adaptation in terms of the continuously varying material heterogeneity. Rather than the local adaptive meshing algorithm proposed in [9], the global adaptive meshing algorithm equalizes the material composition variation within each element and thus reduce the number of element to a great extent in terms of a given material threshold [9]. In addition, the use of ODT greatly improves the mesh quality.

In the remaining of this paper, we describe the algorithm details of the adaptive meshing strategy in Section 2. In Section 3, two examples are used to demonstrate the efficiency of the proposed adaptive meshing method. Finally, conclusion is drawn in Section 4.

2. ADAPTIVE MESHING STRATEGY

In adaptive meshes of homogeneous objects, denser elements are usually generated in the boundary area to conform to the geometric resolution, while coarser elements are created in the interior to save computational resources. However, for heterogeneous objects where two or more different material ingredients exist, this observation is not true anymore. Besides the geometric resolution, the length scale associated with the material heterogeneity in each finite element also has a great impact on the accuracy of FEA solutions [20]. In other words, the adaptive meshing of heterogeneous models is determined by both the material heterogeneity and the geometric complexity. Regarding that, we propose an adaptive tetrahedral mesh generation method, which naturally includes a three-step meshing process: the first step is initial mesh generation, the latter one is material-oriented refinement and the third one is geometry-oriented refinement.

Given a 3D heterogeneous model, an initial mesh is firstly constructed by using the Delaunay-based tetrahedral mesh generation method. The material-oriented refinement is then executed on the initial mesh. In this refinement, a global mesh adaptation method based on ODT is applied. The refinement process is recursively processed until the mesh satisfies a predefined material threshold. Next, a geometry-oriented refinement is further processed in order to tackle the geometric facilities of the heterogeneous model. This refinement process terminates when a set geometric resolution is satisfied. Fig. 2 shows the high-level pseudo-code of this adaptive meshing strategy, the great details of which will be presented in the following subsections.

Adaptive meshing strategy
Input: Heterogeneous model (Section 2.1.1) and a set of meshing criteria (Section 2.1.2)
Initialize a coarse mesh (Section 2.2).
If the material threshold is not met **do**
Material-oriented refinement (Section 2.3)
If the geometric resolution is not met **do**
Geometry-oriented refinement (Section 2.4)

Fig. 2: The pseudo-code of the adaptive meshing strategy for 3D heterogeneous objects.

2.1. Input

The input to the adaptive meshing algorithm includes two parts: heterogeneous model and meshing criteria.

2.1.1. Heterogeneous model

In this paper, the heterogeneous model that we mainly investigate is the analytic heterogeneous model [26], which has been widely utilized in heterogeneous solid modeling. In an analytic heterogeneous model, the geometry of the domain is represented by boundary representation (B-Rep) and the material distribution is represented with explicit analytic functions. Given an arbitrary point (x, y, z) in the Cartesian coordinate system, for instance, the volume fraction of the ith material ingredient at this point can be written as $V_i = f_i(x, y, z)$ [26]. Accordingly, the material compositions at this point can be represented as a vector of volume fractions, i.e.

$$M = [V_1, V_2, \ldots, V_k], 0 \le V_i \le 1, \sum_{i=1}^{k} V_i = 1 \tag{2.1}$$

where k denotes the number of material constituents of the analytic heterogeneous model and the volume fractions are constrained to sum up to unity. In this paper, we mainly focus on the two-phase heterogeneous models ($k=2$) that are often studied in the literature and utilized in real applications. As a result, it is sufficient to use the volume fraction of one alternative material constituent to represent the overall material distribution because of the unity property. In the following sections, by material composition function, we mean the volume fraction function of one individual material constituent.

2.1.2. Meshing criteria

As mentioned above, the adaptive meshing process on heterogeneous objects is determined by the material distribution as well as the geometric complexity, the meshing criteria are therefore composed of a material threshold and a geometric resolution (or approximation error). Since the latter one has been widely described in the literature, we mainly introduce the material threshold in what follows.

We consider an element T is bad when it fails to satisfy a predefined material threshold, δ_0, i.e. $\delta(T) > \delta_0$. Here, $\delta(T)$ denotes the material composition variation within the element T. In this paper, the definition of $\delta(T)$ is consistent with the one in [9], thus we have

$$\delta(T) = \max(\delta_{ij}), \forall ij, i \ne j, j \in 1, 2, \ldots, m \tag{2.2}$$

where m is the number of sample points (e.g. the vertices of T) and δ_{ij} is the magnitude of material composition difference between two arbitrary sample points, P_i and P_j, i.e.

$$\delta_{ij} = \|M_i - M_j\| \tag{2.3}$$

where M_i denotes the material compositions of P_i.

2.2. Initial Mesh Generation

The initial mesh (*I-Mesh*) is constructed by Delaunay-based methods [21,24]. The *I-Mesh* is a Delaunay mesh of all the feature vertices (corners) of the input domain, the minimal number of sample points on surface patches and sharp edges (ceases) [24], and a small set of interior points if necessary. In this step, no material heterogeneity information is involved.

2.3. Material-oriented Refinement

Since the *I-Mesh* is only a coarse mesh and far away from desirable, the material-oriented refinement is consequently executed, aiming to generate a mesh that is validated in terms of the material threshold, δ_0. In this step, a global refinement algorithm based on ODT is proposed to control the mesh adaptation. To keep this paper as self-contained as possible, we first briefly introduce the concept of ODT.

2.3.1. *Optimal Delaunay triangulation*

An ODT is a triangulation that minimizes the interpolation error for the isotropic function $||x||^2$ with a given set of vertices. The concept of ODT is first proposed by Chen and Xu [7], based on which, Chen [6] then developed a mesh smoothing technique to improve the quality of triangular mesh. Alliez et al. [1] and Tournois et al. [24], in addition, extended Chen's method to 3D cases, proposing high-quality tetrahedral mesh generation methods.

In ODT based meshing methods, the high-quality meshes are achieved by minimizing the following quadratic energy [1]:

$$E_{ODT} = \frac{1}{n+1} \sum_{i=1..N} \int_{\Omega_i} \|\mathbf{x} - \mathbf{x}_i\|^2 d\mathbf{x} \tag{2.4}$$

where n is the dimension of the domain of interest, N is the number of vertices, and Ω_i is the 1-ring neighborhood of vertex $\mathbf{x}_i$ [1]. In the process of energy minimization, each vertex is recursively moved to an optimal position within its 1-ring neighborhood [1], i.e.

$$\mathbf{x}^* = \frac{1}{|\Omega_i|} \sum_{T_j \in \Omega_i} |T_j| \mathbf{c}_j \tag{2.5}$$

where $\mathbf{c}_j$ is the circumcenter of the tetrahedron (triangle in 2D cases) T_j, $|T_j|$ and $|\Omega_i|$ are the volume (area in 2D cases) of T_j and Ω_i, respectively. More generally, the above equation for 3D cases can be written as:

$$\mathbf{x}^* = \frac{1}{\sum_{T_k \in \Omega_i} \frac{|T_k|}{\mu^3(\mathbf{g}_k)}} \sum_{T_j \in \Omega_i} \frac{|T_j|}{\mu^3(\mathbf{g}_j)} \mathbf{c}_j \tag{2.6}$$

where μ is a sizing field, and $\mu(\mathbf{g}_j)$ approximately represents the locally desired size of the tetrahedron T_j, in which $\mathbf{g}_j$ is the centroid of T_j.

The merits of ODT-based mesh generation methods are that they guarantee high mesh quality and can easily control the mesh gradation corresponding to a sizing field μ. Inspired by these properties, we creatively apply the concept of ODT into the field of adaptive meshing of heterogeneous models by connecting the sizing filed μ to the material composition functions.

2.3.2. *The sizing field related to the material composition function*

In adaptive meshing of heterogeneous models, we aim to generate denser elements in the field where material composition changes fast, and create relatively coarse elements in the area where material composition changes slowly. In light of that, the sizing field should be inversely proportional to the material composition changing rate (or the gradient of material composition function), i.e.

$$\mu = \frac{C}{|\nabla f|} \tag{2.7}$$

where ∇f is the gradient of the material composition function f, and C is a constant related to the material threshold, δ_0.

2.3.3. *Global refinement algorithm*

Having established the sizing field associated with the material distributions, we now illustrate the global refinement algorithm based on ODT. Fig. 3 shows the flowchart of this algorithm, to which several important steps are listed as below.

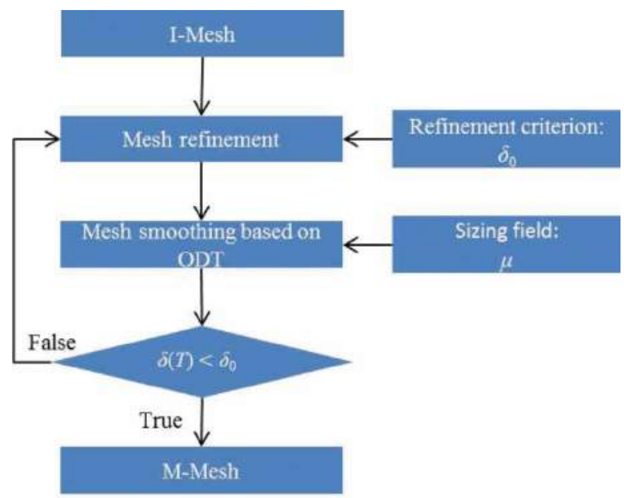

Fig. 3: The flowchart of the material-oriented refinement.

(i) Apply a mesh refinement onto the *I-Mesh*, inserting a Steiner point into every tetrahedron that does not satisfy a set material threshold, δ_0.
(ii) Optimize the mesh obtained in Step (i) by using the ODT-based mesh smoothing coupled with the sizing field μ shown in Eqn. (2.7).
(iii) Evaluate the validity of the mesh generated in Step (ii) in terms of the material threshold, δ_0.
(iv) If the material composition variation of an arbitrary tetrahedron T is lower than the material threshold, δ_0, terminate the refinement algorithm and output the obtained mesh (or *M-Mesh*); otherwise, return to step (i).

It is not difficult to notice that the mesh smoothing based on ODT is iteratively executed in the global refinement process. Through this way, the material composition variations over elements are distributed as equally as possible. Meanwhile, high-quality meshes are guaranteed.

2.4. Geometry-oriented Refinement

Notably, only material heterogeneity information is taken into account in the material-oriented refinement. This might cause problems whenever coarse elements are generated near curved boundaries of the domain where finer elements are expected to account for the geometric facilities. To solve this problem, the geometry-oriented refinement is applied. The geometry-oriented refinement (see Fig. 4) is guided by a surface or crease approximation error [3]. A Delaunay refinement [21,22] is firstly processed on the *M-Mesh*, followed by a Laplacian mesh smoothing [4,11]. It should be noted that only those newly inserted vertices are allowed to move during the Laplacian meshing smoothing. Otherwise, the material variation within tetrahedra will be changed and might exceed the material threshold. We call the finally obtained mesh in this step, *G-Mesh*.

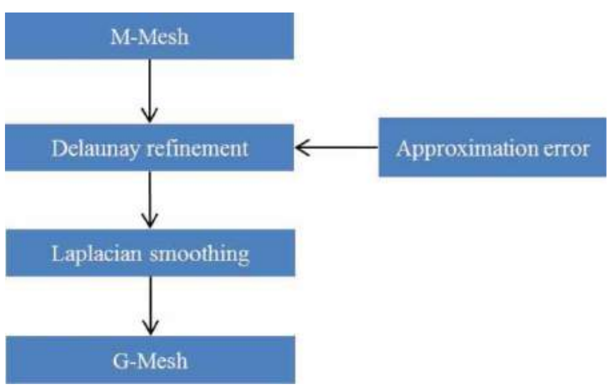

Fig. 4: The flowchart of geometry-oriented refinement.

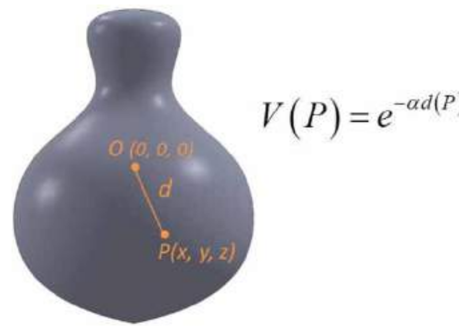

Fig. 5: The geometric description and the material composition function of an analytic heterogeneous model in case study I. Here *d* is the Euclidean distance from an arbitrary point *P* to the reference point *O*, and α is a constant coefficient.

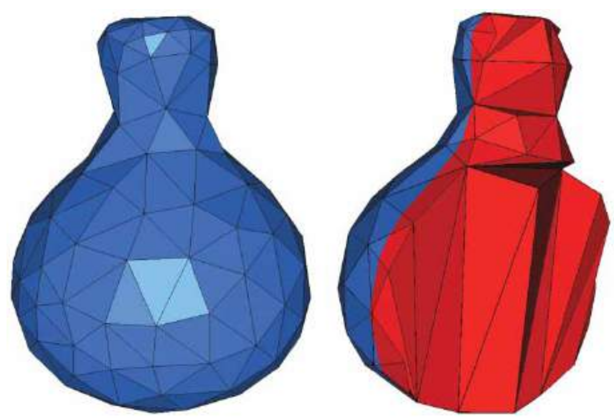

Fig. 6: The *I-Mesh* generated on the analytic heterogeneous model of case study I. Left: view of the *I-Mesh*; right: cut view of the *I-Mesh*.

3. IMPLEMENTATION DETAILS

We have implemented the proposed algorithm successfully by using the non-commercial library, CGAL [5]. In this section, two case studies are presented to show the efficacy of the proposed adaptive tetrahedral mesh generation method.

3.1. Case study I: Analytic Heterogeneous Model with Exponential Function-based Material Distribution

Figure 5 depicts the heterogeneous model of case study I, in which the material composition function is an exponential function. Fig. 6 shows the *I-Mesh* generated on this heterogeneous model. Since a limited number of vertices are inserted, this mesh is far from desirable and needs further refinement.

Fig. 7 shows the *M-Mesh* after the material-oriented refinement with respect to a set material threshold, $\delta_0 = 0.1$. Notice that denser elements are generated in the center of the domain where material composition changes fast (see the volume fraction function in Fig. 5) and relatively coarse elements are created in boundary area where material composition changes slowly. In addition, this mesh satisfies the material threshold and the material composition variation of its most elements obviously concentrates to the interval [0.05, 0.1] (see the left top of Fig. 7). We can also note from Tab. 1 that the average material composition variation is 0.0724, which is very close to the material threshold 0.1. In this sense, the material-oriented refinement algorithm indeed distributes the material composition variation as equally as possible. Except the mesh adaptation, high-quality mesh is also guaranteed since all the dihedral angles are constrained into the interval [10.91, 163.80] (see the left bottom of Fig. 7).

From Fig. 7, we can observe that the *M-Mesh* does not approximate the geometry of the heterogeneous model well for coarse elements are generated in the boundary area. To solve such a problem, the geometry-oriented refinement is further applied. Fig. 8 shows the *G-Mesh* after the geometry-oriented

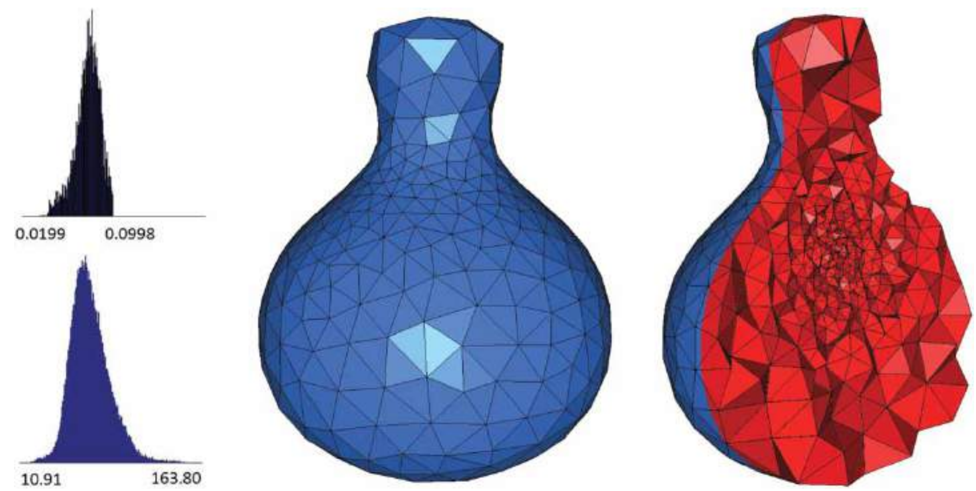

Fig. 7: The *M-Mesh* generated on the analytic heterogeneous model of case study I. Left top: the distribution of material composition variation over the *M-Mesh*; left bottom: the distribution of dihedral angles over the *M-Mesh*; middle: view of the *M-Mesh*; right: cut view of the *M-Mesh*.

				Material Variation			*Dihedral Angle*	
	N_p	N_f	N_t	*Min.*	*Max.*	*Avg.*	*Min.*	*Max.*
I-Mesh	160	632	447					
M-Mesh	2616	856	14356	0.0199	0.0998	0.0724	10.91	163.80
G-Mesh	14840	16430	63911	0.0020	0.0998	0.0301	10.01	164.07

Tab. 1: Statistics related to the meshes depicted in Fig. 6, Fig. 7 and Fig. 8. Here N_p, N_f, and N_t denote the number of vertices, facets, and tetrahedra respectively.

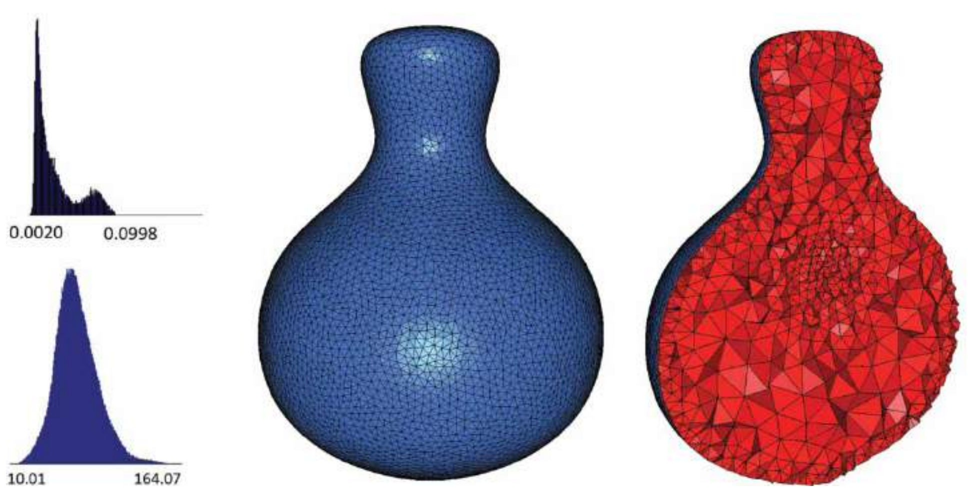

Fig. 8: The *G-Mesh* generated on the analytic heterogeneous model of case study I. Left top: the distribution of material composition variation over the *G-Mesh*; left bottom: the distribution of dihedral angles over the *G-Mesh*; middle: view of the *G-Mesh*; right: cut view of the *G-Mesh*.

refinement. Notice that finer elements are constructed in the boundary area to conform to the geometric resolution. Although the mesh has been revised, the distribution of material composition variation still satisfies the material threshold (see the left top of Fig. 8) since the vertices that belong to the *M-Mesh* are not allowed to move during the geometry-oriented refinement. As a large number of elements are generated in the boundary area where the material composition varies slowly, the average material composition variation decreases from 0.0724 to 0.0301 (see Tab. 1)

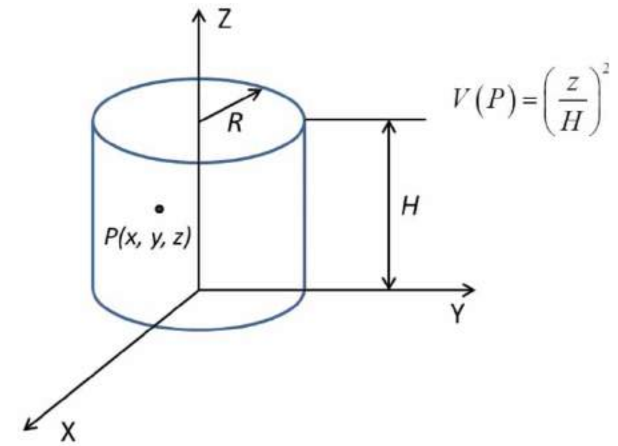

Fig. 9: The geometric description and the material composition function of the analytic heterogeneous model in case study II. Here H and R are the height and the radius of the cylinder, respectively.

3.2. Case study II: Analytic Heterogeneous Model with Power Function-based Material Distribution

Figure 9 depicts the analytic heterogeneous model of case study II. Different from case study I, the material composition of this heterogeneous model is a power function. Other than this, the implementation details of the adaptive meshing algorithm are the same in both cases, therefore we only show here the final mesh, *G-Mesh*, for the sake of simplicity.

Figure 10 shows the *G-Mesh* with respect to a set material threshold $\delta_0 = 0.1$. Notice that denser elements are generated in the top area where material composition changes fast (refer to the volume fraction function in Fig. 9), as well as in the boundary area to precisely approximate the geometric facilities. We can also note that the material composition variations are distributed as equally as possible over the elements since most of them concentrate to the interval [0.05, 0.1] (see the left top of Fig. 10). In addition, high-quality mesh is achieved for all dihedral angles are constrained into the interval [15.07, 157.16] (see the left bottom of Fig. 10).

4. CONCLUSION

The contribution of this study includes:

- Developing an adaptive tetrahedral mesh generation method of 3D FGMs.
- Demonstrating a step-wise meshing strategy to control the mesh adaptation in terms of the material heterogeneity and the geometric complexity, respectively.
- Introducing a global mesh adaptation algorithm based on ODT to tackle the continuously varying material heterogeneity. Rather than those local adaptive meshing algorithms, the global adaptive meshing algorithm equalizes the material composition variation within each element and thus reduces the number of element to a great extent in terms of a given material threshold.
- Providing great potentials for accurate and efficient FEA simulations of heterogeneous materials.

One weakness of this study is that only analytic heterogeneous models are investigated. In the future, we are going to extend the proposed method to other more complex heterogeneous models, such as the heterogeneous feature tree (HFT) based model [15], in which the heterogeneous material distributions cannot be directly represented by analytic functions.

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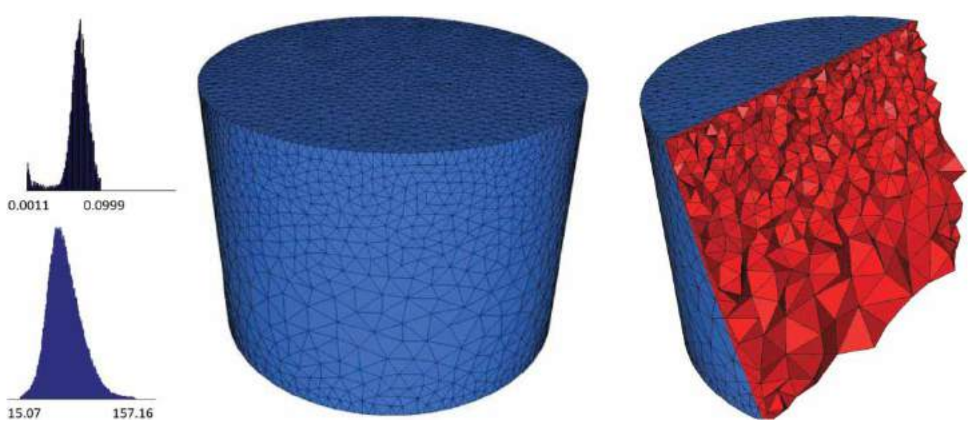

Fig. 10: The *G-Mesh* generated on the analytic heterogeneous model. Left top: the distribution of material composition variation over the *G-Mesh*; left bottom: the distribution of dihedral angles over the *G-Mesh*.

REFERENCES

[1] Alliez, P.; Cohen-Steiner, D.; Yvinec, M.; Desbrun, M.: Variational tetrahedral meshing, Acm Transactions on Graphics, 24(3), 2005, 617–625. http://dx.doi.org/10.1145/1073204.1073238

[2] Anlas, G.; Santare, M. H.; Lambros, J.: Numerical calculation of stress intensity factors in functionally graded materials, International Journal of Fracture, 104(2), 2000, 131–143. http://dx.doi.org/10.1023/A:1007652711735

[3] Boissonnat, J.-D.; Oudot, S.: Provably good sampling and meshing of surfaces, Graphical Models, 67(5), 2005, 405–451. http://dx.doi.org/10.1016/j.gmod.2005.01.004

[4] Canann, S. A.; Tristano, J. R.; Staten, M. L.: An Approach to Combined Laplacian and Optimization-Based Smoothing for Triangular, Quadrilateral, and Quad-Dominant Meshes in IMR, 1998.

[5] CGAL. Available from: https://www.cgal.org/.

[6] Chen, L.: Mesh Smoothing Schemes Based on Optimal Delaunay Triangulations in Proceedings, 13th International Meshing Roundtable, Williamsburg, VA, Sandia National Laboratories, 2004.

[7] Chen, L.; Xu, J. C.: Optimal Delaunay triangulations, Journal of Computational Mathematics, 22(2), 2004, 299–308.

[8] Chi, S. H.; Chung, Y. L.: Mechanical behavior of functionally graded material plates under transverse load - Part II: Numerical results, International Journal of Solids and Structures, 43(13), 2006, 3675–3691. http://dx.doi.org/10.1016/j.ijsolstr.2005.04.010

[9] Chiu, W. K.; Kou, X. Y.; Tan, S. T.: Adaptive meshing of 2D heterogeneous objects using material quadtree, Computer-Aided Design & Applications, 8(2), 2011, 289–300. http://dx.doi.org/10.3722/cadaps.2011.289-300

[10] Cuillière, J.-C.; Francois, V.; Drouet, J.-M.: Automatic 3D Mesh Generation of Multiple Domains for Topology Optimization Methods in Proceedings of the 21st International Meshing Roundtable, 2013. http://dx.doi.org/10.1007/978-3-642-33573-0_15

[11] Freitag, L. A.: On combining Laplacian and optimization-based mesh smoothing techniques, ASME APPLIED MECHANICS DIVISION-PUBLICATIONS-AMD, 220, 1997, 37–44.

[12] Giunta, G.; Belouettar, S.; Carrera, E.: Analysis of FGM Beams by Means of Classical and Advanced Theories, Mechanics of Advanced Materials and Structures, 17(8), 2010, 622–635. http://dx.doi.org/10.1080/15376494.2010.518930

[13] Kou, X. Y.; Parks, G. T.; Tan, S. T.: Optimal design of functionally graded materials using a procedural model and particle swarm optimization, Computer-Aided Design, 44(4), 2012, 300–310. http://dx.doi.org/10.1016/j.cad.2011.10.007

[14] Kou, X. Y.; Tan, S. T.: Heterogeneous object modeling: A review, Computer-Aided Design, 39(4), 2007, 284–301. http://dx.doi.org/10.1016/j.cad.2006.12.007

[15] Kou, X. Y.; Tan, S. T.: A hierarchical representation for heterogeneous object modeling, Computer-Aided Design, 37(3), 2005, 307–319. http://dx.doi.org/10.1016/j.cad.2004.03.006

[16] Kou, X. Y.; Tan, S. T.: A systematic approach for integrated computer-aided design and finite element analysis of functionally-graded-material objects, Materials & Design, 28(10), 2007, 2549–2565. http://dx.doi.org/10.1016/j.matdes.2006.10.024

[17] Kumar, V.; Burns, D.; Dutta, D.; Hoffmann, C.: A framework for object modeling, Computer-Aided Design, 31(9), 1999, 541. http://dx.doi.org/10.1016/S0010-4485(99)00051-2

[18] Miyamoto, Y.; Kaysser, W. A.; Rabin, B. H.; Kawasaki, A.; Ford, R. G.: Functionally graded materials: design, processing, and applications, Kluwer Academic Publishers, Boston, 330 p, 1999. http://dx.doi.org/10.1007/978-1-4615-5301-4

[19] Pasko, A.; Adzhiev, V.; Comninos, P.: Heterogeneous Objects Modelling and Applications, Vol. 4889, Springer Berlin Heidelberg, 2008. http://dx.doi.org/10.1007/978-3-540-68443-5

[20] Paulino, G. H.; Kim, J.-H.: The weak patch test for nonhomogeneous materials modeled with graded finite elements, Journal of the Brazilian Society of Mechanical Sciences and Engineering, 29, 2007, 63–81. http://dx.doi.org/10.1590/S1678-58782007000100010

[21] Rineau, L.; Yvinec, M.: Meshing 3D Domains Bounded by Piecewise Smooth Surfaces* in Proceedings of the 16th international meshing roundtable, 2008. http://dx.doi.org/10.1007/978-3-540-75103-8_25

[22] Shewchuk, J. R.: Tetrahedral mesh generation by Delaunay refinement in Proceedings of the fourteenth annual symposium on Computational geometry, Minneapolis, Minnesota, USA, 1998. http://dx.doi.org/10.1145/276884.276894

[23] Shin, K. H.: Adaptive mesh generation for finite element analysis of functionally graded materials in American Society of Mechanical Engineers, Computers and Information in Engineering Division, CED, Orlando, FL, 2005. http://dx.doi.org/10.1115/IMECE2005-82442

[24] Tournois, J.; Wormser, C.; Alliez, P.; Desbrun, M.: Interleaving Delaunay Refinement and Optimization for Practical Isotropic Tetrahedron Mesh Generation, Acm Transactions on Graphics, 28(3), 2009. http://dx.doi.org/10.1145/1531326.1531381

[25] Zhang, Y. J.; Hughes, T. J. R.; Bajaj, C. L.: An automatic 3D mesh generation method for domains with multiple materials, Computer Methods in Applied Mechanics and Engineering, 199(5–8), 2010, 405–415. http://dx.doi.org/10.1016/j.cma.2009.06.007

[26] Zhu, F.: Visualized CAD modeling and layered manufacturing modeling for components made of a multiphase perfect material, PhD Thesis, University of Hong Kong, 2004. http://dx.doi.org/10.5353/th_b3007384