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# An Extension Algorithm for Disk B-Spline Curve with $G^2$ Continuity

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**ABSTRACT**

Curve extension is a useful function in CAD systems. Disk B-Spline curve has its distinct advantages in representing a 2D region. This paper presents an algorithm for extending the disk B-Spline curve with $G^2$ continuity. A disk Bezier segment is used to construct the extending part and $G^2$-continuity can be used to describe the smoothness at the joint disk. Fairness of the extending disk Bezier curve segment is achieved by minimizing energy objective functions for the center curve and the radius function separately. New control disks are computed by unclamping algorithm to represent the whole extended disk B-Spline curve. The experimental results demonstrate the effectiveness of our method.

**Keywords:** curve extension, disk B-Spline curve, $G^2$ continuity, minimal energy

## 1. INTRODUCTION

Disk B-Spline curve (DBSC) has its distinct advantages in representing a 2D region. It represents not only the boundary but also every point in the region, allows flexible manipulation like deformation and morphing, and requires smaller datasets. In 2004 and 2005, Wu and Seah et al. [17][12] first proposed the disk B-Spline curve through extending disk Bezier curve [4], for representing artistic brushstrokes and applications in 2D animation. In the representation, various attributes such as scalar and vector fields can be applied to the stroke and the intermediate animation frames can be automatically generated linearly or non-linearly. Later, DBSC based brushstroke representation was applied to interactive 2D free-hand drawings, which were able to achieve real-time and visual effects for arts and industrial design applications [13]. After that, the DBSC based stroke representation started attracting attentions from other researchers. Xie et al. [18][19] used the DBSC based representation in their interactive sketch-based system. Nijboer et al. [9][10] referred to the DBSC based stroke representation in the stroke representation of their sketch system. Melikhov et al. [6][7] used DBSC in line drawings. DBSC was also applied to in-betweening [1] and auto-coloring [11] in 2D animation. Cheng et al. [2] used DBSC in shape blending.

Curve extension is a general problem in curve design. A given B-Spline curve may need to be extended in order to meet some geometric shape conditions or engineering requirements. A natural problem is to extend the original B-Spline curve to the given point, and to represent the extended curve in B-Spline form. Shetty [14] proposed a practical and straightforward method for extending rational B-Spline curves without modifying the shape and parameterization of the original curves. Hu et al. [3] proposed a curve unclamping algorithm for B-Spline extensions. The original curve and the extending segment satisfy $C^2$-continuity at the joint point. However, the extended curve is exclusive and in some cases is not a desired result. Zhou et al. [21] used Bezier curve to construct the B-Spline curve, extending segment with $G^2$-continuity to describe the smoothness at the joint point, and established objective functions based on minimum energy and minimum curvature variation to determine the degrees of freedom of the extending curve.

As for disk B-Spline curve extension, Zhang et al. [20] proposed a DBSC extension algorithm based on



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curve unclamping. With their method, the unclamping knot vector of the extended DBSC is computed according to the accumulated chord length method. The original DBSC and the extending disk Bezier curve segment are $C^2$-continuous at the joint disk. But the extended DBSC cannot be adjusted manually and is not desired in some large torsion cases. Geometric continuity plays an important role in depicting curves' fairness and can supply additional degrees of freedom for adjustment. In this paper, we propose a $G^2$-continuity extension algorithm for DBSC. The shape of the extended DBSC is determined by minimizing strain energy of the disk curve. The unclamping knot vector is determined by the degree of freedom at which the minimal energy of the center curve of the extending disk curve is achieved. New control disks are computed to represent the whole extended DBSC. Our method is important for DBSC modeling in 2D animation and many other areas.

The rest of the paper is organized as follows. The definition of disk B-Spline curve and the $G^2$-continuity conditions for DBSCs are described in section 2. Our DBSC extension algorithm is presented in section 3. Some experimental examples and comparisons are shown in section 4. We draw conclusions in section 5.

## 2. $G^2$-CONTINUITY CONDITIONS FOR DISK B-SPLINE CURVES

### 2.1. Definition of Disk B-Spline Curve

A p-degree disk B-Spline curve is defined as:

$$\langle D\rangle(t) = \sum_{i=0}^{n} N_{i,p}(t)\langle P_i; r_i\rangle$$

where $P_i$ is control point and $r_i$ is control radius. $\langle P_i; r_i\rangle$ is a disk in the plane defined as $\langle P; r\rangle = \{x \in R^2 \,||x - P| \le r, P \in R^2, r \in R^+\}$. $N_{i,p}(t)$ is the p-degree B-Spline basis function defined over the knot vector $T = [t_0, \cdots, t_{p+1}, \cdots, t_n, \cdots, t_{n+p+1}]$.

As $\langle D\rangle(t) = \sum_{i=0}^{n} N_{i,p}(t)\langle P_i; r_i\rangle = \sum_{i=0}^{n}\langle N_{i,p}(t)P_i; N_{i,p}(t)r_i\rangle = \langle \sum_{i=0}^{n} N_{i,p}(t)P_i; \sum_{i=0}^{n} N_{i,p}(t)r_i\rangle$, a DBSC can be viewed as two parts: the center curve $P(t) = \sum_{i=0}^{n} N_{i,p}(t)P_i$, which is a B-Spline curve and the radius function $r(t) = \sum_{i=0}^{n} N_{i,p}(t)r_i$, which is a B-Spline scalar function. Therefore most of the properties and algorithms of DBSC can be obtained by applying B-Spline curve and function to the two parts of DBSC respectively [17].

### 2.2. $G^2$-continuity Conditions

Suppose two DBSCs $\langle D\rangle(t) = \sum_{i=0}^{n} N_{i,p}(t)\langle P_i; r_i\rangle$ and $\langle \overline{D}\rangle(t) = \sum_{i=0}^{n} N_{i,p}(t)\langle \overline{P_i}; \overline{r_i}\rangle$ with a joint disk in between. According to the $G^2$-continuity conditions for B-Spline curves [15], the $G^2$-continuity conditions at the joint disk for the two DBSCs can be defined as follows:

$$\begin{cases} \langle \overline{D}\rangle(0) = \langle D\rangle(1) \\ \langle \overline{D}\rangle'(0) = \alpha\langle D\rangle'(1) \\ \langle \overline{D}\rangle''(0) = \alpha^2\langle D\rangle''(1) + \beta\langle D\rangle'(1) \end{cases} \quad (2.1)$$

where $\alpha > 0$ and $\beta$ is an arbitrary real number.

Eqn. (2.1) is the unified form of the $G^2$-continuity conditions for DBSCs. As the center curve describes the skeleton of the DBSCs and the radius function describes the width of the DBSCs, the center curve and radius function are two independent attributes of DBSCs. So we can discuss the $G^2$-continuity conditions for the center curve and the radius function of the DBSCs separately, in order to acquire more fairness of the DBSCs.

For the $G^2$-continuity conditions of the center curve, we have

$$\begin{cases} \bar{P}(0) = P(1) \\ \bar{P}'(0) = \alpha_1 P'(1) \\ \bar{P}''(0) = {\alpha_1}^2 P''(1) + \beta_1 P'(1) \end{cases} \quad (2.2)$$

where $\alpha_1 > 0$ and $\beta_1$ is an arbitrary real number.

For the $G^2$-continuity conditions of the radius function, we acquire

$$\begin{cases} \bar{r}(0) = r(1) \\ \bar{r}'(0) = \alpha_2 r'(1) \\ \bar{r}''(0) = {\alpha_2}^2 r''(1) + \beta_2 r'(1) \end{cases} \quad (2.3)$$

where $\alpha_2 > 0$ and $\beta_2$ is an arbitrary real number. Eqn. (2.3) is the scalar form of the $G^2$-continuity conditions of the center curve in Eqn. (2.2).

## 3. DISK B-SPLINE CURVE EXTENSION WITH $G^2$-CONTINUITY

### 3.1. Extending Principle

Given a cubic disk B-Spline curve $\langle D\rangle(t) = \sum_{i=0}^{n} N_{i,3}(t)\langle P_i; r_i\rangle$ with clamped knot vector $T = [0, 0, 0, 0, t_4, \ldots t_n, 1, 1, 1, 1]$. $\langle Q; R\rangle$ is the given extending disk at center $Q$ with radius $R$. We construct the extending curve segment in cubic disk Bezier form $\langle B\rangle(u) = \sum_{i=0}^{3} B_i^3(u)\langle Q_i; R_i\rangle$ [4], where $B_i^3(u)$ is the cubic Bernstein basis function over $[0, 1]$. A cubic disk Bezier curve can also be regarded as two parts: the center curve $Q(u) = \sum_{i=0}^{3} B_i^3(u)Q_i$, which is a cubic Bezier curve; the radius function $R(u) = \sum_{i=0}^{3} B_i^3(u)R_i$, which is a cubic Bezier scalar function.

The original DBSC $\langle D\rangle(t)$ and the extending disk Bezier curve $\langle B\rangle(u)$ satisfy $G^2$-continuity at the extending disk. Applying the $G^2$-continuity conditions for

DBSCs in Eqn. (2.1), we can get

$$\begin{cases} \langle B\rangle(0) = \langle D\rangle(1) \\ \langle B\rangle'(0) = \alpha\langle D\rangle'(1) \\ \langle B\rangle''(0) = \alpha^2\langle D\rangle''(1) + \beta\langle D\rangle'(1) \end{cases} \tag{3.1}$$

Here we discuss the $G^2$-continuity conditions for the center curve and the radius function separately. As for the center curve, we acquire

$$\begin{cases} Q(0) = P(1) \\ Q'(0) = \alpha_1 P'(1) \\ Q''(0) = {\alpha_1}^2 P''(1) + \beta_1 P'(1) \end{cases} \tag{3.2}$$

where $\alpha_1 > 0$ and $\beta_1$ is an arbitrary real number.

As for the radius function, we acquire

$$\begin{cases} R(0) = r(1) \\ R'(0) = \alpha_2 r'(1) \\ R''(0) = {\alpha_2}^2 r''(1) + \beta_2 r'(1) \end{cases} \tag{3.3}$$

where $\alpha_2 > 0$ and $\beta_2$ is an arbitrary real number.

To obtain a simple solution, we make $\beta_1 = 0$ and $\beta_2 = 0$, and leave degree of freedom $\alpha_1$ for center curve adjustment and degree of freedom $\alpha_2$ for radius function adjustment. Then we expand Eqn. (3.2) and Eqn. (3.3). The control disks $\langle Q_i; R_i\rangle(i = 0, 1, 2, 3)$ of the extending disk Bezier curve segment $\langle B\rangle(u)$ can be expressed as follows:

$$\begin{cases} Q_0 = P_n \\ Q_1 = P_n + \alpha_1 \frac{P_n - P_{n-1}}{t_{n+3} - t_n} \\ Q_2 = P_n \\ +2\alpha_1 \frac{P_n - P_{n-1}}{t_{n+3} - t_n} + \alpha_1^2 \left(\frac{P_n - P_{n-1}}{(t_{n+3} - t_n)(t_{n+2} - t_n)} - \frac{P_{n-1} - P_{n-2}}{(t_{n+2} - t_{n-1})(t_{n+2} - t_n)}\right) \\ Q_3 = Q \end{cases} \tag{3.4}$$

$$\begin{cases} R_0 = r_n \\ R_1 = r_n + \alpha_2 \frac{r_n - r_{n-1}}{t_{n+3} - t_n} \\ R_2 = r_n + 2\alpha_2 \frac{r_n - r_{n-1}}{t_{n+3} - t_n} \\ +\alpha_2^2 \left(\frac{r_n - r_{n-1}}{(t_{n+3} - t_n)(t_{n+2} - t_n)} - \frac{r_{n-1} - r_{n-2}}{(t_{n+2} - t_{n-1})(t_{n+2} - t_n)}\right) \\ R_3 = R \end{cases}$$

Through minimizing energy objective functions for the center curve and the radius function separately, the degrees of freedom $\alpha_1$ and $\alpha_2$ for the two parts can be determined. The extending disk Bezier curve can be expressed. Then the unclamping knot vector of the extended DBSC can be specified. After reparameterization, the original DBSC and the extending disk Bezier curve satisfy $C^2$-continuity at the joint disk. New control disks are computed by unclamping algorithm to represent the whole extended DBSC.

### 3.2. Center Curve Fairness by Minimal Energy

In the extending principle to determine the center curve, we have one degree of freedom $\alpha_1$ for adjustment. We can achieve more fairness of the extending disk Bezier curve $\langle B\rangle(u)$ through minimizing energy objective function of the center Bezier curve to determine the degree of freedom $\alpha_1$. Here we choose the exact energy variation [21] as the objective function of the center curve.

$$E_{curve} = \int k^2(s)ds \tag{3.5}$$

where $ds$ is differential of curve arc length and $k(s)$ is curvature defined as $k(s) = \|Q'(u) \times Q''(u)\| / \|Q'(u)\|^3$. $Q(u) = \sum_{i=0}^{3} B_i^3(u)Q_i$ is the center curve function of the extending disk Bezier curve.

$$E_{curve} = \int k^2(s)ds = \int_0^1 \frac{(x''(t)y'(t) - x'(t)y'(t))^2}{\left(\sqrt{(x'(t))^2 + (y'(t))^2}\right)^5} dt \tag{3.6}$$

Solving Eqn. (3.6) is a strong non-linear problem. Eqn. (3.6) can be decomposed into sum of square form by composite trapezoidal rule:

$$\int_{t_0}^{t_n} f(t)dt = \frac{t_n - t_0}{n}\left(\sum_{k=1}^{n-1} f\left(t_0 + k\frac{t_n - t_0}{n}\right) + \frac{f(t_0)}{2} + \frac{f(t_n)}{2}\right) \tag{3.7}$$

where $n$ is the number of subintervals.

Let $h_1 = (x''(t)y'(t) - x'(t)y'(t))^2$ and $h_2 = \left(\sqrt{(x'(t))^2 + (y'(t))^2}\right)^5$. Denote $g(t) = \sqrt{h_1/h_2}$. Based on the discrete integral of Eqn. (3.7), objective function Eqn. (3.6) can be rewritten as:

$$E(\alpha_1) = \frac{1}{n}\left(\left(\frac{g(0)}{\sqrt{2}}\right)^2 + \left(\frac{g(1)}{\sqrt{2}}\right)^2 + \sum_{k=1}^{n-1} g^2\left(\frac{k}{n}\right)\right) \tag{3.8}$$

Let

$$A(\alpha_1) = \left(\frac{g(0)}{\sqrt{2}}, \frac{g(1)}{\sqrt{2}}, g\left(\frac{1}{n}\right), \cdots, g\left(\frac{n-1}{n}\right)\right) \tag{3.9}$$

Then minimizing objective function Eqn. (3.6) is equivalent to minimizing Eqn. (3.9), which is a non-linear least-squares problem and can be solved by Gauss-Newton method [21]. The algorithm below gives the iterated process.

Step 1: Choose an initial value $\alpha_0 > 0$.

Step 2: Eqn. (3.9) is approximated using the first order Taylor series expansion:

$$A(\alpha_1) \approx A(\alpha_0) + A'(\alpha_0)(\alpha_1 - \alpha_0).$$

Step 3: Let $M(\alpha_0) = A'(\alpha_0)$, $b(\alpha_0) = A'(\alpha_0)\alpha_0 - A(\alpha_0)$. Calculate $\alpha_1 = (M^T(\alpha_0)M(\alpha_0))^{-1}M^T(\alpha_0)b(\alpha_0)$.

Step 4: If $|\alpha_1 - \alpha_0| \le \varepsilon$, stop the iteration and output $\alpha_1$; else $\alpha_0 = \alpha_1$ and go to Step 2, repeat the process. Here $\varepsilon$ is a given threshold value.

Given a proper $\alpha_0$, the Gauss-Newton algorithm is convergent. So we can determine the optimal degree of freedom $\alpha_1$ and thus the center curve of the extending disk Bezier curve.

### 3.3. Radius Function Fairness by Minimal Energy

In the extending principle to determine the radius function, we have one degree of freedom $\alpha_2$ for adjustment. We can achieve more fairness of the extending disk Bezier curve $\langle B\rangle(u)$ through minimizing energy objective function of the radius function to determine the degree of freedom $\alpha_2$.

#### *3.3.1. The constraints of the degree of freedom $\alpha_2$*

Obviously we have to make sure that the radii of the control disks in $\langle B\rangle(u)$ are positive, that is, $R_1 > 0$ and $R_2 > 0$. Now we discuss the constraints of the degree of freedom $\alpha_2$ in different cases according to the original disk B-Spline curve. Note that $\alpha_2 > 0$.

We make $A = \frac{r_n - r_{n-1}}{(t_{n+3}-t_n)(t_{n+2}-t_n)} - \frac{r_{n-1}-r_{n-2}}{(t_{n+2}-t_{n-1})(t_{n+2}-t_n)}$, $B = \frac{r_n - r_{n-1}}{t_{n+3}-t_n}$, and $C = r_n$. Then we get

$$\begin{cases} R_1 = C + B\alpha_2 > 0 \\ R_2 = C + 2B\alpha_2 + A\alpha_2^2 > 0 \end{cases} \tag{3.10}$$

Supposing $\Delta = (2B)^2 - 4AC$, in different cases of $A$, $B$ and $\Delta$, Tab. 1. provides the constraints of $\alpha_2$.

| $A$ | $B$ | $\Delta$ | The constraints of $\alpha_2$ |
|---|---|---|---|
| $A > 0$ | $B > 0$ | | $\alpha_2 > 0$ |
| | $B < 0$ | $\Delta > 0$ | $0 < \alpha_2 < \frac{-2B - \sqrt{\Delta}}{2A}$ |
| | | $\Delta \le 0$ | $0 < \alpha_2 < -\frac{C}{B}$ |
| | $B = 0$ | $\Delta \ge 0$ | This case doesn't exist. |
| | | $\Delta < 0$ | $\alpha_2 > 0$ |
| $A < 0$ | | $\Delta > 0$ | $0 < \alpha_2 < \frac{-2B - \sqrt{\Delta}}{2A}$ |
| | | $\Delta \le 0$ | This case doesn't exist. |
| $A = 0$ | $B \ge 0$ | | $\alpha_2 > 0$ |
| | $B < 0$ | | $0 < \alpha_2 < -\frac{C}{2B}$ |

Tab. 1: The constraints of degree of freedom $\alpha_2$ in different cases.

#### *3.3.2. Radius function fairness by minimal energy*

We optimize the radius function of the extending disk Bezier curve to determine the degree of freedom $\alpha_2$. Here we choose the second order energy [16] as the objective function of the radius function.

$$E_{radius} = \int_0^1 \|R''(u)\|^2 du \tag{3.11}$$

where $R(u) = \sum_{i=0}^{3} B_i^3(u) R_i$ is the radius function of the extending Bezier segment.

Then Eqn. (3.11) becomes $E(\alpha_2) = \int_0^1 \left\| \sum_{i=0}^{3} B_i^{3''}(u) R_i \right\|^2 du$. It can be shown that $E(\alpha_2)$ is a polynomial of $\alpha_2$ of order four, as

$$E(\alpha_2) = c_4\alpha_2^4 + c_3\alpha_2^3 + c_2\alpha_2^2 + c_1\alpha_2 + c_0 \tag{3.12}$$

where $c_i (i = 0, 1, \ldots 4)$ are all constants. Minimizing function Eqn. (3.12) is equal to solving a cubic equation:

$$E'(\alpha_2) = 4c_4\alpha_2^3 + 3c_3\alpha_2^2 + 2c_2\alpha_2 + c_1 = 0 \tag{3.13}$$

The closed-form solutions of Eqn. (3.13) can be found analytically. Considering the constraints of $\alpha_2$, the monotonicity of the Eqn. (3.12) can be determined. Thus the optimal $\alpha_2$ with minimal energy of the radius function can be determined. The radius function of the extending disk Bezier curve can be achieved.

After achieving the optimal degrees of freedom $\alpha_1$ and $\alpha_2$ for the center curve and radius function of the extending disk Bezier curve separately, the control disks of the disk Bezier curve in Eqn. (3.4) can be determined. Thus the extending disk Bezier curve can be represented. In the following representation of the whole extended DBSC, the optimal degree of freedom $\alpha_1$ is used to determine the unclamping vector of the original DBSC.

### 3.4. Re-computing Control Disks of Disk B-Spline Curve

Through minimizing strain energy of the extending disk Bezier curve, we acquire $1 + \alpha_1$ as the corresponding knot value of $\langle Q; R\rangle$. Let $u = \frac{t-1}{\alpha_1}$. Then the extending disk Bezier curve $\langle B\rangle(u)$ is reparameterized as $\langle \overline{B}\rangle(t) = \langle B\rangle(\frac{t-1}{\alpha_1})$, $t \in [1, 1+\alpha_1]$. $\langle \overline{B}\rangle(t)$ and $\langle D\rangle(t)$ are $C^2$-continuous at t = 1. A whole DBSC can be constructed as follows:

$$\langle D'\rangle(t) = \begin{cases} \langle D\rangle(t), t \in [0, 1] \\ \langle \overline{B}\rangle(t), t \in [1, 1+\alpha_1] \end{cases} \tag{3.14}$$

with unclamped knot vector $T = [0, 0, 0, 0, t_4, \ldots, t_n, 1, 1, 1, 1+\alpha_1, 1+\alpha_1, 1+\alpha_1, 1+\alpha_1]$ and control disks $\{\langle P_0; r_0\rangle, \langle P_1; r_1\rangle, \cdots, \langle P_n; r_n\rangle, \langle Q_1; R_1\rangle, \langle Q_2; R_2\rangle, \langle Q; R\rangle\}$ [8].

The multiplicity of '1' in the above knot vector can be reduced, and a DBSC unclamping algorithm is used to re-compute the new control disks for the whole DBSC [5].

The whole extended DBSC can be presented as

$$\langle D'\rangle(t) = \sum_{i=0}^{n+1} N_{i,p}(t) \langle P_i'; r_i'\rangle \tag{3.15}$$

where control disks $\langle P_{n+1}'; r_{n+1}'\rangle = \langle Q; R\rangle$, and $\langle P_i'; r_i'\rangle$ $(i = 0, 1, \cdots, n)$ are determined as follows. For the cubic DBSC, $p = 3$.

(1) Set $P_j^{-1} = P_j$, $r_j^{-1} = r_j$, $j = n-p+1, \ldots, n$

$$(2)\quad P_j^i = \begin{cases} P_j^{i-1}, j = n-p+1,\ldots,n-i+1 \\ \dfrac{P_j^{i-1} - (1-\alpha_{i,j})P_{j-1}^i}{\alpha_{i,j}}, j = n-i,\ldots,n \end{cases}$$

$$r_j^i = \begin{cases} r_j^{i-1}, j = n-p+1,\ldots,n-i+1 \\ \dfrac{r_j^{i-1} - (1-\alpha_{i,j})r_{j-1}^i}{\alpha_{i,j}}, j = n-i,\ldots,n \end{cases}$$

$$\alpha_{i,j} = \frac{t_{n+1} - t_j}{t_{i+j+2} - t_j}, i = 0,1,\ldots,p-2$$

$$(3)\quad P_j' = \begin{cases} P_j, j = 0,1,\ldots,n-p \\ P_j^{p-2}, j = n-p+1,\ldots,n \end{cases}$$

$$r_j' = \begin{cases} r_j, j = 0,1,\ldots,n-p \\ r_j^{p-2}, j = n-p+1,\ldots,n \end{cases}$$

With the DBSC unclamping algorithm above, the representation of the whole extended DBSC is achieved. Note that the knot vector can be normalized without a change of geometry as

$$T = \left[0,0,0,0,\frac{t_4}{1+\alpha_1},\ldots,\frac{t_n}{1+\alpha_1},\frac{1}{1+\alpha_1},1,1,1,1\right].$$

### 3.5. Extension with Multiple Target Disks

We generalize the method for extending one cubic DBSC to two target disks. For a given DBSC $\langle D\rangle(t) = \sum_{i=0}^{n} N_{i,3}(t)\langle P_i; r_i\rangle$, two disk Bezier curves $\langle B_1\rangle(u)$and $\langle B_2\rangle(u)$ are constructed as the extending curve segment to the two target disks $\langle Q_1; R_1\rangle$and $\langle Q_2; R_2\rangle$ respectively. Suppose $\langle B_1\rangle(u)$ is constructed using the above method, and the optimal degrees of freedom $\alpha_{11}$ and $\alpha_{12}$ for the center curve and radius function of $\langle B_1\rangle(u)$ are obtained. Thus the corresponding knot value of $\langle Q_1; R_1\rangle$ is determined as $1+\alpha_{11}$. With the DBSC unclamping algorithm, the first extended DBSC for extending $\langle D\rangle(t)$ to $\langle Q_1; R_1\rangle$ is represented, denoted by $\langle D_1\rangle(t)$. $\langle B_2\rangle(u)$ can be constructed in the same way through extending $\langle D_1\rangle(t)$ to $\langle Q_2; R_2\rangle$. The optimal degrees of freedom $\alpha_{21}$ and $\alpha_{22}$ for the center curve and radius function of $\langle B_2\rangle(u)$ can be obtained. Then the corresponding knot value of $\langle Q_2; R_2\rangle$ is calculated as $1+\alpha_{11}+\alpha_{21}$. With the DBSC unclamping algorithm, the second extended DBSC for extending $\langle D_1\rangle(t)$ to $\langle Q_2; R_2\rangle$ is represented, denoted by $\langle D_2\rangle(t)$. In this way $\langle D_2\rangle(t)$ is the extended DBSC for extending DBSC $\langle D\rangle(t)$ to two given disks $\langle Q_1; R_1\rangle$and $\langle Q_2; R_2\rangle$. Case of more than two extending disks is similar.

## 4. EXPERIMENTAL RESULTS

In this section, we give experimental results and provide comparisons of our method with the accumulated chord length parameterization method proposed in [20]. The difference between these two methods is the determination of the unclamping vector of the extended DBSC. In our proposed method, $G^2$ continuity offers degrees of freedom for the extending curve segment and we use the degree of freedom acquired through minimal energy method to determine the unclamping vector of the extended DBSC. Method in [20] uses accumulated chord length parameterization method to determine the unclamping vector of the extended DBSC, in which way only exclusive extending result exists.

We can compare these two methods through visual and numeric comparisons. The strain energy and rotation number of the center curve of the extended DBSC are two indices for numeric comparisons. Strain energy is defined as $E_{curve} = \int k^2(s)ds$, which is explained in detail in section 3.2. Rotation number is an important tool for analyzing the whole shape property of curve in differential geometry. It describes the rotation of tangent or normal vector moving along a curve. For a planar curve$P(t)(0 < t < 1)$, rotation number is defined as $Rot = \frac{1}{2\pi}\int_0^1 |k(t)|\|P'(t)\|dt$, where $k(t)$ is the curvature of $P(t)$[8].

Fig. 1 shows the first example of extending DBSC to one target disk. Fig. 1(a) shows the given DBSC and the extending disk. Fig. 1(b) gives the extension result based on the accumulated chord length parameterization method. Fig. 1(c) shows the $G^2$-continuity extension result of our minimal energy method with optimal degrees of freedom $\alpha_1 = 0.327$ and $\alpha_2 = 0.375$.

Fig. 2 shows the second example of extending DBSC to one target disk. Fig. 2(a) shows the given DBSC and the extending disk. Fig. 2(b) gives the extension result based on the accumulated chord length parameterization method. Fig. 2(c) shows the $G^2$-continuity extension result of our minimal energy method with optimal degrees of freedom $\alpha_1 = 0.278$ and $\alpha_2 = 0.0156$.

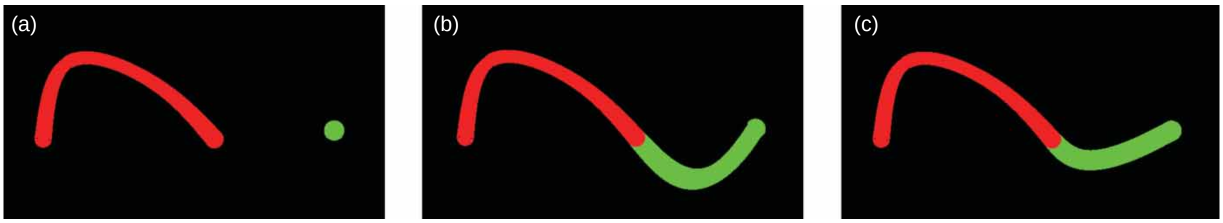


Fig. 1: Extension results of DBSC to one target disk. (a) The given DBSC and the target disk; (b) Extension by accumulated chord length parameterization method; (c) Extension by our minimal energy method.

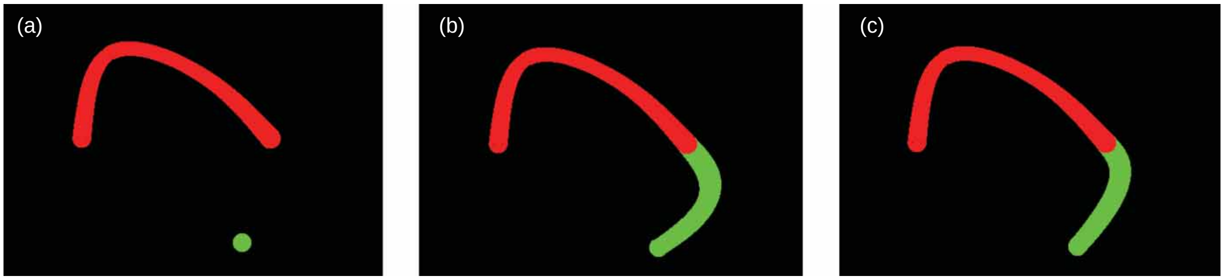


Fig. 2: Extension results of DBSC to one target disk. (a) The given DBSC and the target disk; (b) Extension by accumulated chord length parameterization method; (c) Extension by our minimal energy method.

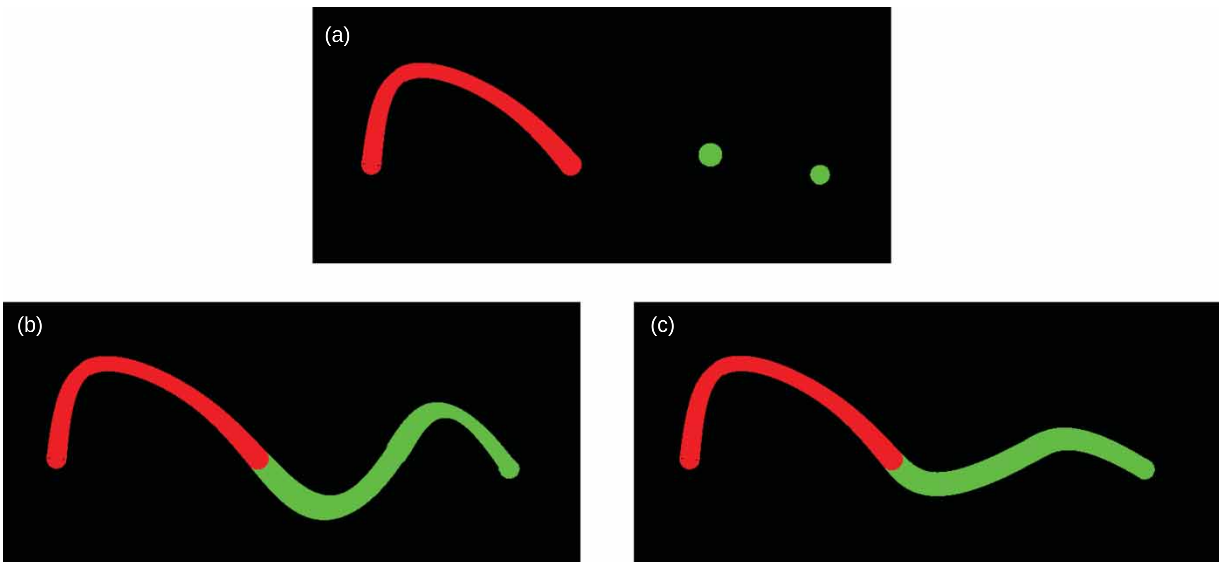


Fig. 3: Extension results of DBSC to two target disks. (a) The given DBSC and the target disks; (b) Extension by accumulated chord length parameterization method; (c) Extension by our minimal energy method.

| | Fig. 1 | | Fig. 2 | | Fig. 3 | |
|---|---|---|---|---|---|---|
| | Curve (b) | Curve (c) | Curve (b) | Curve (c) | Curve (b) | Curve (c) |
| Energy | 0.02897 | 0.01961 | 0.03209 | 0.02779 | 0.29750 | 0.13346 |
| Rot | 0.29475 | 0.21608 | 0.27422 | 0.23154 | 0.51153 | 0.30407 |

Tab. 2: Numeric comparisons of different experimental examples.

Fig. 3 shows an example of extending DBSC to two target disks. Fig. 3(a) shows the given DBSC and the two extending disks. The extension result based on the accumulated chord length parameterization method is shown in Fig. 3(b). Fig. 3(c) shows the $G^2$-continuity extension result of our minimal energy method with optimal degrees of freedom $\alpha_{11} = 0.327$, $\alpha_{12} = 0.375$, $\alpha_{21} = 0.045$ and $\alpha_{22} = 0.084$.

From visual effects, these two methods can both satisfy the visual continuity at the joint disk, while the curve length of extended DBSC using the accumulated chord length parameterization method is longer than the one using our minimal energy method, which means more energy consumption.

Tab. 2 gives the numeric comparisons of the experimental examples by these two methods. From the computed results, the extended DBSCs generated by our minimal energy method have less strain energy and less rotation number than the ones by accumulated chord length parameterization method. Comparisons show that our method can acquire better extension results.

## 5. CONCLUSIONS

In this paper an extension algorithm for disk B-Spline curve with $G^2$ continuity is presented. The extending segment is expressed in disk Bezier curve and a whole extended disk B-Spline curve is represented. $G^2$-continuity is used to describe the smoothness of the joint disk and the extending disk Bezier curve fairness is achieved by minimizing energy objectives for the center curve and radius function respectively. The experimental results verify the effectiveness of our method. This work can lead to wider and further applications of DBSC in 2D region modeling.

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