Quasi-Log-Aesthetic Curves in Polynomial Bézier Form

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This paper presents planar quasi-log-aesthetic curves in polynomial Bézier form. Log-aesthetic curves are curves that can be considered as the generalization of the Clothoid, Nielsen's spiral, logarithmic spirals and circle involute. By deriving the Taylor polynomials of log-aesthetic curves and converting the basis to Bernstein basis, we obtain quasi-log-aesthetic curves in polynomial Bézier form. We show the implementation results with logarithmic curvature graphs and a G^1 Hermite interpolation method.

Keywords: log-aesthetic curves, Bézier curves, logarithmic curvature graphs.
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1 INTRODUCTION

Planar log-aesthetic curves [3,4,5,8] are curves with linear logarithmic curvature graphs (LCGs)[12]. The curves were originally proposed for the shape design of highly aesthetic objects. However, log-aesthetic curves are only computed by integrating the equations and they are not compatible with freeform curves such as Bézier, B-spline or NURBS curves. This paper proposes a method for approximating log-aesthetic curves in terms of polynomial Bézier curves. Thus as the degree gets higher, the curve gets closer to log-aesthetic curves.

There are several related works for approximating some special cases of log-aesthetic curves by free-form curves. Baumgarten and Farin proposed a method for approximating logarithmic spirals (log-aesthetic curves with $\alpha = 1$) by rational cubic Bézier curves [1]. Wang et al. have presented a method for approximating the Clothoid curve (log-aesthetic curve with $\alpha = -1$) by polynomial Bézier curves using Taylor series [7]. Both of these methods have only dealt with the specific cases log-aesthetic curves.

Yoshida and Saito have proposed a method for approximating log-aesthetic curves by rational cubic Bézier curves [10]. They showed that the linearity of the logarithmic curvature graph is well-preserved by the approximation except when log-aesthetic curves include nearby point of $\rho = 0$ or ∞. Miura et al. have proposed a method for approximating log-aesthetic curves with polynomial curves by discritizing the equations of log-aesthetic curves [6]. Miura's work was motivated by class A Bézier curves [2] proposed by Farin. Yoshida et al. have found that typical class A Bézier curves gets closer to a logarithmic spiral, which is a log-aesthetic curve with $\alpha = 1$, as the degree gets higher [9]. Miura et al. have generalized this to log-aesthetic curves.

This paper proposes quasi-log-aesthetic curves in polynomial Bézier form and compares them with the curves generated by Miura's method [6] and quasi-log-aesthetic curves in rational cubic Bézier form [10]. We show that quasi-log-aesthetic curves are better approximation than the curves generated by Miura's method. We also propose a method for G^1 Hermite interpolation by specifying two endpoints and their tangents..

2 REVIEW OF PLANAR LOG-AESTHETIC CURVES

Log-aesthetic curves are curves that can be considered as the generalization of the Clothoid, the logarithmic spiral, the circle involute, and a circle. Harada et al. have originally proposed log-aesthetic curves [3,4]. The curves were originally called aesthetic curves but now the name of log-aesthetic curves is used. The word "aesthetic" is used because log-aesthetic curves are based on Harada's analysis of many aesthetic curve segments of the natural and artificial objects. Harada et al. have shown that the logarithmic curvature graphs(LCGs) of these aesthetic curves segments can be approximated by straight lines. The curves with linear logarithmic curvature graphs are called log-aesthetic curves. The linearity of LCG guarantees that the curvature is monotonically varying.

Logarithmic curvature graphs are graphs whose horizontal and vertical axes are $\log\rho$ and $\log(\rho\, \mathrm{d}s/\mathrm{d}\rho)$, respectively.12]. The slope of the LCG is called α, which determines the shape of the curve. When $\alpha=-1,0,1,2$ or $\pm\infty$, the log-aesthetic curve becomes the Clothoid, Nielsen's spiral, logarithmic spiral, circle involute, or circle, respectively.

Log-aesthetic curves can be represented in terms of tangential angle or arc length. For the derivation of the equations, refer to [8]. The equation of log-aesthetic curves $\mathbf{P}(\theta)$ in terms of tangential angle θ is

$$\mathbf{P}(\theta)=\begin{bmatrix}P_x(\theta)\\ P_y(\theta)\end{bmatrix}=\begin{bmatrix}\int_0^{\theta}\rho(\varphi)\cos\varphi \mathrm{d}\varphi\\ \int_0^{\theta}\rho(\varphi)\sin\varphi \mathrm{d}\varphi,\end{bmatrix}, \tag{1}$$

where

$$\rho(\varphi)=\begin{cases} e^{\Lambda\varphi} & \text{if } \alpha=1,\\ \left((\alpha-1)\Lambda\varphi+1\right)^{\frac{1}{\alpha-1}} & \text{otherwise.}\end{cases} \tag{2}$$

Here, $\rho(\varphi)$ is the radius of curvature at the tangential angle φ. Λ is the parameter that performs the similarity transformation when $\alpha\neq1$. When $\alpha=1$, changing Λ changes the shape of the curves. See [8]. Note that Eqn. (1) (and also Eqn. (3)) is the standard form [8]. This means that a certain point of the curve, which is determined by Λ, is translated to the origin, rotated so that the tangent becomes $[1 \quad 0]^{\mathrm{T}}$ and scaled so that the radius of curvature becomes 1 at the origin.

The equation of log-aesthetic curves $\mathbf{Q}(s)$ in terms of tangential angle in the standard form is

$$\mathbf{Q}(s)=\begin{bmatrix}Q_x(s)\\ Q_y(s)\end{bmatrix}=\begin{bmatrix}\int_0^{s}\cos(\theta(u))\mathrm{du}\\ \int_0^{s}\sin(\theta(u))\mathrm{du}\end{bmatrix}, \tag{3}$$

where

$$\theta(u)=\begin{cases}\dfrac{1-e^{-\Lambda u}}{\Lambda} & \text{if } \alpha=0\\ \dfrac{\log(\Lambda u+1)}{\Lambda} & \text{if } \alpha=1\\ \dfrac{(\Lambda\alpha u+1)^{\left(1-\frac{1}{\alpha}\right)}-1}{\Lambda(\alpha-1)} & \text{otherwise.}\end{cases} \tag{4}$$

Note that Eqn. (1) and Eqn. (3) represent the same curve. Tangential angle θ and arc length s are related by

$$s=\begin{cases}-\dfrac{\log(1-\Lambda\theta)}{\Lambda} & \text{if } \alpha=0\\ \dfrac{e^{\Lambda\theta}-1}{\Lambda} & \text{if } \alpha=1\\ \dfrac{(1+(\alpha-1)\Lambda\theta)^{\frac{\alpha}{\alpha-1}}-1}{\Lambda\alpha} & \text{otherwise.}\end{cases} \tag{5}$$

When we work with log-aesthetic curves, we have to be careful that s and θ may have bounds depending on α and Λ. See [8] for the detail of the bounds.

3 QUASI-LOG-AESTHETIC CURVES IN POLYNOMIAL BÉZIER FORM

To represent log-aesthetic curves in polynomial Bézier form, we will use the Taylor series of log-aesthetic curves up to the user-specified degree n. Given a function $f(x)$, the Taylor polynomials of degree n about $x=a$ is

$$\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n \tag{6}$$

As was shown in Section 2, the equation of log-aesthetic curves can be represented either by the function of tangential angle or by the function of arc length. We will first derive the Taylor polynomial of log-aesthetic curves in terms of tangential angle. Let $g_x(\theta)=\rho(\theta)\cos\theta, g_y(\theta)=\rho(\theta)\sin\theta$, where $\rho(\theta)$ is defined in Eqn. (2). Using Eqn. (6), the Taylor polynomials of Eqn. (1) about θ_a is

$$\mathbf{P}_{Taylor}(\theta)=\begin{bmatrix}\int_0^{\theta_a} g_x(\varphi)\mathrm{d}\varphi+g_x(\theta_a)(\theta-\theta_a)+\dfrac{g_x^{(1)}(\theta_a)}{2!}(\theta-\theta_a)^2+\cdots+\dfrac{g_x^{(n-1)}(\theta_a)}{n!}(\theta-\theta_a)^n\\ \int_0^{\theta_a} g_y(\varphi)\mathrm{d}\varphi+g_y(\theta_a)(\theta-\theta_a)+\dfrac{g_y^{(1)}(\theta_a)}{2!}(\theta-\theta_a)^2+\cdots+\dfrac{g_x^{(n-1)}(\theta_a)}{n!}(\theta-\theta_a)^n\end{bmatrix}, \tag{7}$$

where $g_x^{(n)}(\theta_a)$ and $g_y^{(n)}(\theta_a)$ are the n-th derivatives of $g_x(\theta)$ and $g_y(\theta)$ evaluated at θ_a, respectively. Since $\mathbf{P}_{Taylor}(\theta)$ is a polynomial of degree n, replacing θ with t and converting the power basis to Bernstein basis, we can get a Bézier curve $\mathbf{P}_\theta(t)$:

$$\mathbf{P}_\theta(t)=\sum_{i=0}^{n}\mathbf{b}_i B_i^n(t) \tag{8}$$

where

$$B_i^n(t)=\binom{n}{i}t^i(1-t)^{n-i}\ , \tag{9}$$

and $\mathbf{b}_i$ are the control points computed from Eqn. (7) by the basis conversion. We will call Eqn. (8) quasi-log-aesthetic curve in polynomial Bézier form in terms of tangential angle.

The Taylor polynomials in terms of arc length can be derived in the similar manner. Let $h_x(s)=\cos(\theta(s)), h_y(s)=\sin(\theta(s))$, where $\theta(s)$ is define in Eqn. (4). The Taylor polynomials of Eqn. (3) about s_a is

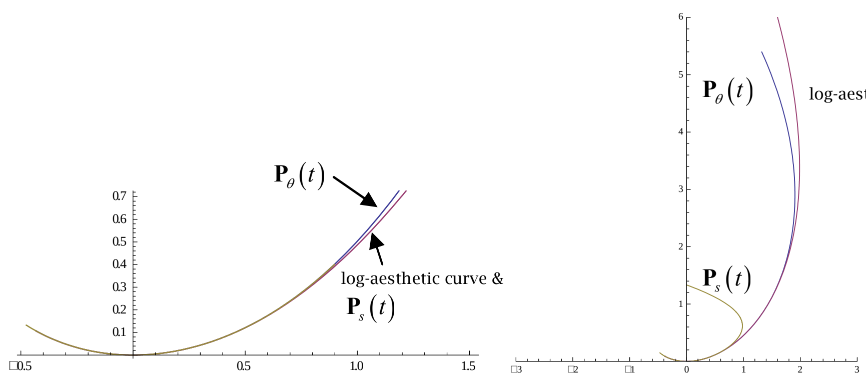
$$\mathbf{Q}_{Taylor}(s)=\begin{bmatrix} \int_0^{s_a} h_x(u)\mathrm{d}u+h_x(s_a)(s-s_a)+\frac{h_x^{(1)}(s_a)}{2!}(s-s_a)^2+\cdots+\frac{h_x^{(n-1)}(s_a)}{n!}(s-s_a)^n \\ \int_0^{s_a} h_y(u)\mathrm{d}u+h_y(s_a)(s-s_a)+\frac{h_y^{(1)}(s_a)}{2!}(s-s_a)^2+\cdots+\frac{h_x^{(n-1)}(s_a)}{n!}(s-s_a)^n \end{bmatrix}, \tag{10}$$

where $h_x^{(n)}(s_a)$ and $h_y^{(n)}(s_a)$ are the n-th derivatives of $h_x(s)$ and $h_y(s)$ evaluated at s_a, respectively. Replacing s of $\mathbf{Q}_{Taylor}(s)$ with t and changing to the Bernstein basis, we get quasi-log-aesthetic curve in polynomial Bézier form in terms of arc length:

$$\mathbf{P}_s(t)=\sum_{i=0}^{n}\mathbf{b}_i B_i^n(t). \tag{11}$$

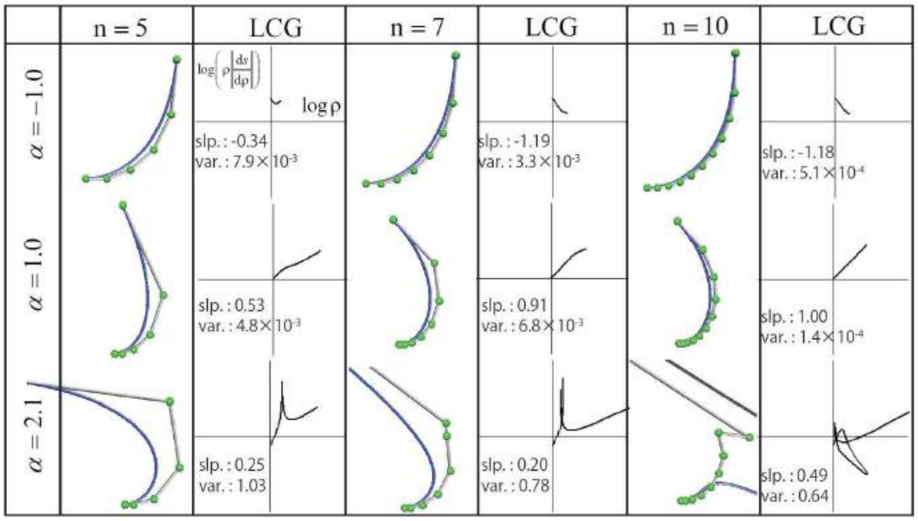
Fig. 1 shows log-aesthetic curves and quasi-log-aesthetic curves $\mathbf{P}_\theta(t)$ and $\mathbf{P}_s(t)$ of degree 5 about $\theta_a=0$. In Fig. 1(a) where $\alpha=-1, \Lambda=0.5$, log-aesthetic curves and $\mathbf{P}_s(t)$ is almost the same and $\mathbf{P}_\theta(t)$ is close to the log-aesthetic curve. In Fig. 1(b) where $\alpha=-1, \Lambda=0.5$, we can see that $\mathbf{P}_\theta(t)$ is a better approximation of log-aesthetic curves than $\mathbf{P}_s(t)$. In this research, we use quasi-log-aesthetic curves $\mathbf{P}_\theta(t)$ (Eqn. (7)).

Eqn. (7) is dependent on θ_a. To see how the approximate curve behaves, we have changed θ_a. Fig. 2 shows the quasi-log-aesthetic curves with $\theta_a=0$ (Fig. 2(a)) and $\theta_a=1$ (Fig. 2b)) and their LCGs. In the figure, "slp" means the slope of the LCG computed by the least squares and "var" means the variance. As Fig. 2 shows, when $\alpha<1$, the linearity of the LCGs of $\theta_a=0$ is better than the linearity of the LCGs of $\theta_a=1$. When $\alpha\geq 1$, the linearity of the LCGs of $\theta_a=1$ is better. Thus we use $\theta_a=0$ when $\alpha<1$ and $\theta_a=1$ when $\alpha\geq 1$.

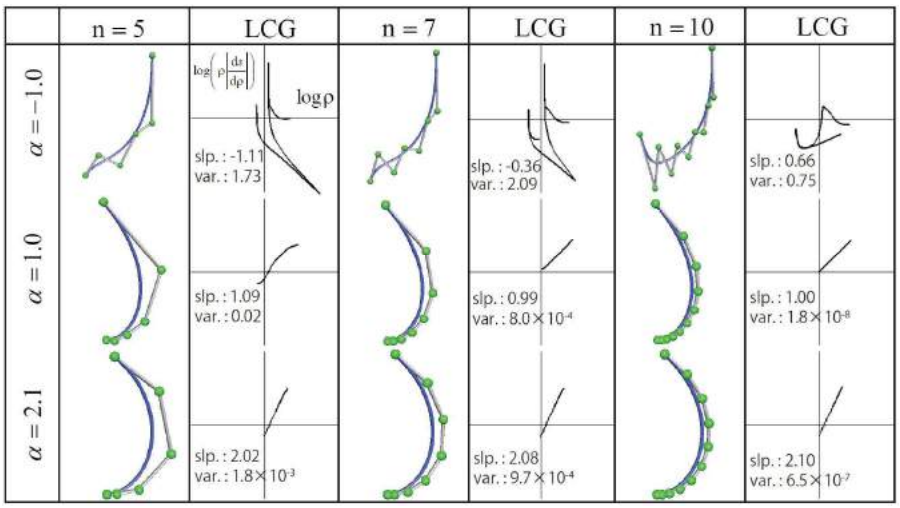


(a) $\alpha=-1, \Lambda=0.5$ (b) $\alpha=1, \Lambda=1$

Fig. 1: Log-aesthetic curves and quasi-log-aesthetic curves $\mathbf{P}_\theta(t)$ and $\mathbf{P}_s(t)$.



(a) Taylor polynomials of $\theta_a = 0$



(b) Taylor polynomials of $\theta_a = 1$

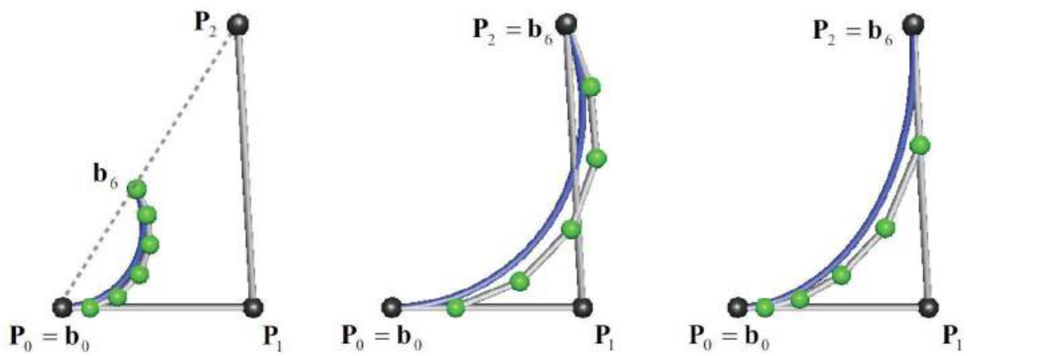
Fig. 2: Changing θ_a.

4 G^1 HERMITE INTERPOLATION

This section presents a G^1 Hermite interpolation method of quasi-log-aesthetic curves. We are given the slope of the LCG α, the degree of the Bézier curve, two endpoints $\mathbf{P}_0, \mathbf{P}_2$ and a point $\mathbf{P}_1$. The tangential directions at $\mathbf{P}_0$ and $\mathbf{P}_2$ are determined by $\mathbf{v}_0 = (\mathbf{P}_1 - \mathbf{P}_0)/|\mathbf{P}_1 - \mathbf{P}_0|$ and $\mathbf{v}_1 = (\mathbf{P}_2 - \mathbf{P}_1)/|\mathbf{P}_2 - \mathbf{P}_1|$, respectively. Let $\mathbf{b}_i (i = 0,\ldots,n)$ be Bézier control points and $\Delta\mathbf{b}_i = \mathbf{b}_{i+1} - \mathbf{b}_i$. We assume that quasi-log-aesthetic curves are represented by tangential angle. Our method can be easily applied to quasi-log-aesthetic curves in terms of arc length by using Eqn. (5).

If we are given α, Λ and θ_a, we can get the equation of a quasi-log-aesthetic curve. In case of log-aesthetic curves, we can easily know the tangential direction at any point of the curve. But in Eqn. (6), we do not know the tangential direction until we draw the curve because parameter t does not exactly correspond to θ. Thus we cannot directly apply the G^1 Hermite interpolation algorithm described in [8].

The G^1 Hermite interpolating quasi-log-aesthetic curve can be found in the following algorithm. Suppose that Λ is known. We set $t_0 = 0$, translate the curve so that $\mathbf{P}_\theta(t_0)$ goes to $\mathbf{P}_0$ and rotate the curve so that the tangent at $\mathbf{P}_\theta(t_0)$ is toward $\mathbf{P}_1 - \mathbf{P}_0$. Now the positional and tangential constraints at the start point are satisfied. Then we will find the parameter t_1 such that $\mathbf{P}_\theta(t_1)$ is on the line that goes through $\mathbf{P}_0$ and $\mathbf{P}_2$ (Fig.3(a)). This process can be considered as the 2D version of the cone intersection method [11]. By scaling the curve by a factor of $|\mathbf{P}_2 - \mathbf{P}_0| / |\mathbf{P}_\theta(t_1) - \mathbf{P}_\theta(t_0)|$ with the scaling center placed at $\mathbf{P}_0$, the endpoint constraints at $\mathbf{P}_2$ is satisfied(Fig. 3(b)). The final process of is changing Λ as was described in [10] so that the tangential constraint at $\mathbf{P}_2$ is satisfied(Fig. 3(c)). Now we can find the G^1 Hermite interpolating quasi-log-aesthetic curve. Note that each time Λ is changed, we need to recompute t_1 and scale the curve. Note also that Λ satisfying the tangential constraint at $\mathbf{P}_2$ may not be found similarly as in log-aesthetic curves [8]. In such a case, the curve segment is not drawn.



(a) t_1 is determined (b) positional coincidence at $\mathbf{P}_2$ (c) tangential coincidence at $\mathbf{P}_2$

Fig. 3: The endpoint and the tangential coincidence.

5 RESULTS

We compare quasi-log-aesthetic curves with Bézier curves generated by Miura's method [6] using logarithmic curvature graphs. Miura's method discretizes log-aesthetic curves, and then using the discretized equation, Bézier control points are placed on the curve either by a constant arc length or by a constant tangential angle.

Figs. 4-7 show Bézier curve segments that approximate log-aesthetic curves. In Fig. 4, 5, 6 and 7, log-aesthetic curves with $\alpha = -1$, 0, 1 and 2, respectively, are approximated. In all of the figures, the degrees of Bézier curves are 6, 10 and 20 from left to right, and red curves are log-aesthetic curves. Figs. 4(a), 5(a), 6(a) and 7(a) show Bézier curve segments generated by the discretization with a constant arc length. Figs. 4(b), 5(b), 6(b) and 7(b) show Bézier curve segments generated by the discretization with a constant tangential angle. Fig. 4(c), 5(c), 6(c) and 7(c) show quasi-log-aesthetic curves. In all the cases, as the degree of Bézier curves gets higher, the curve gets close to the log-aesthetic curve.

To the right of the curves, the LCGs of approximated Bézier curves are shown. "slp" is the slope of the LCG computed by least squares. "var" means the variance from the line of the LCG computed by least squares. For example, in Fig. 4 where the log-aesthetic curve with $\alpha = -1$ is approximated, the slope of the LCG and the variance should be close to -1 and 0, respectively. In all of the method, the slopes of LCGs get closer to the original α as the degree gets higher.

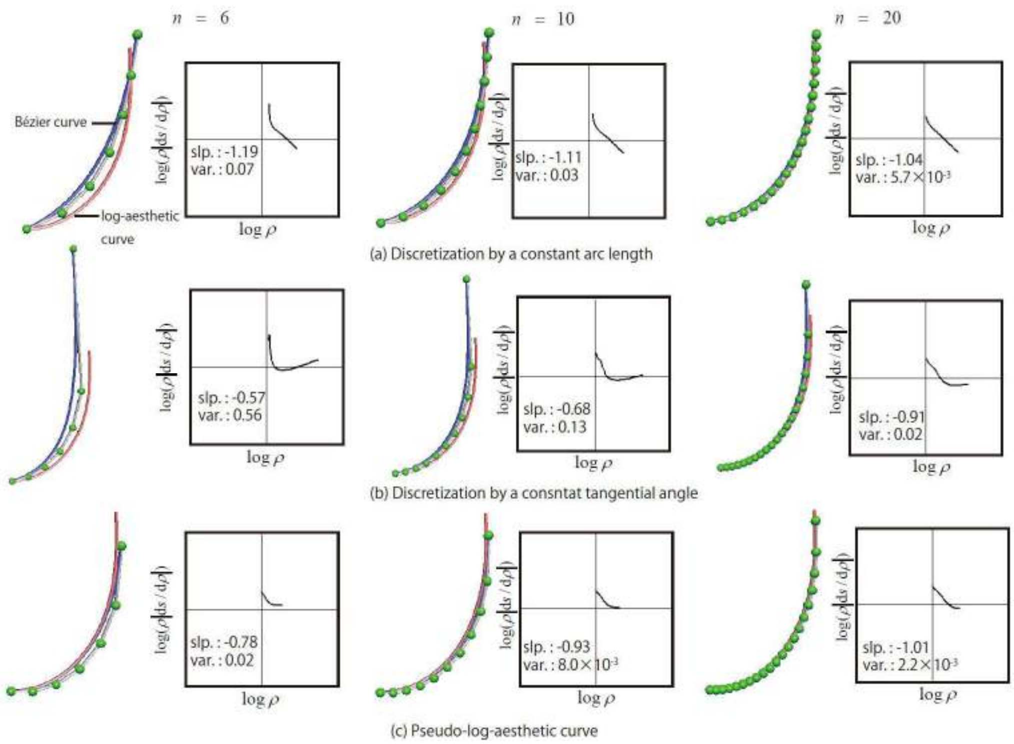


Fig. 4: Approximated log-aesthetic curves ($\alpha = -1$).

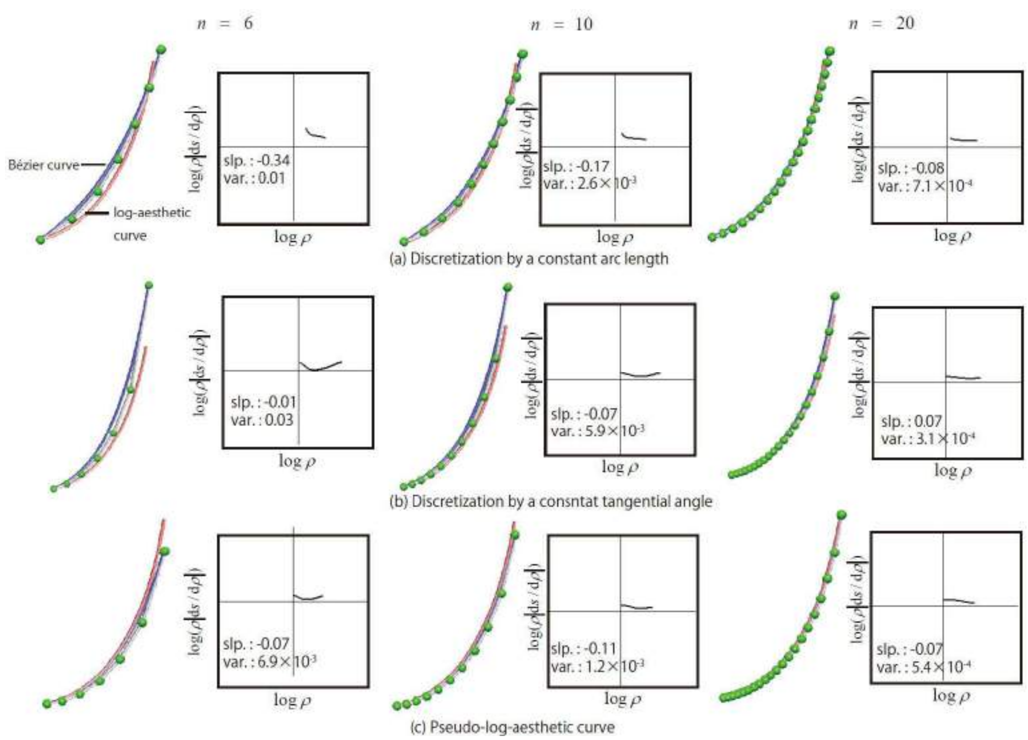


Fig. 5: Approximated log-aesthetic curves ($\alpha = 0$).

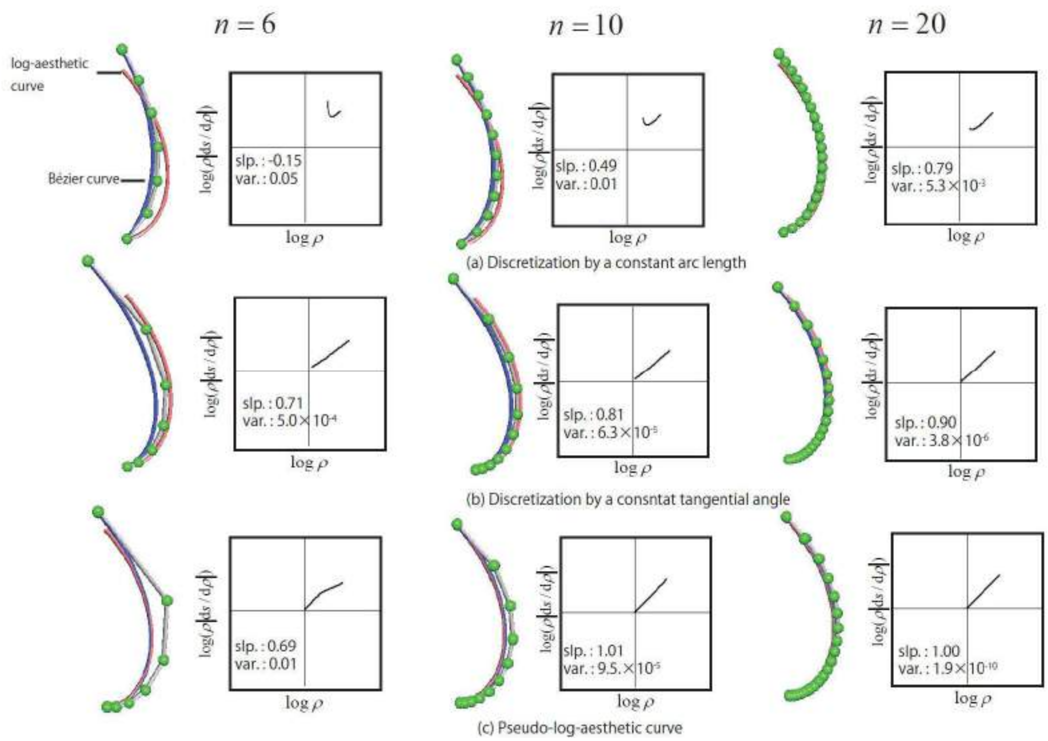


Fig. 6: Approximated log-aesthetic curves ($\alpha = 1$).

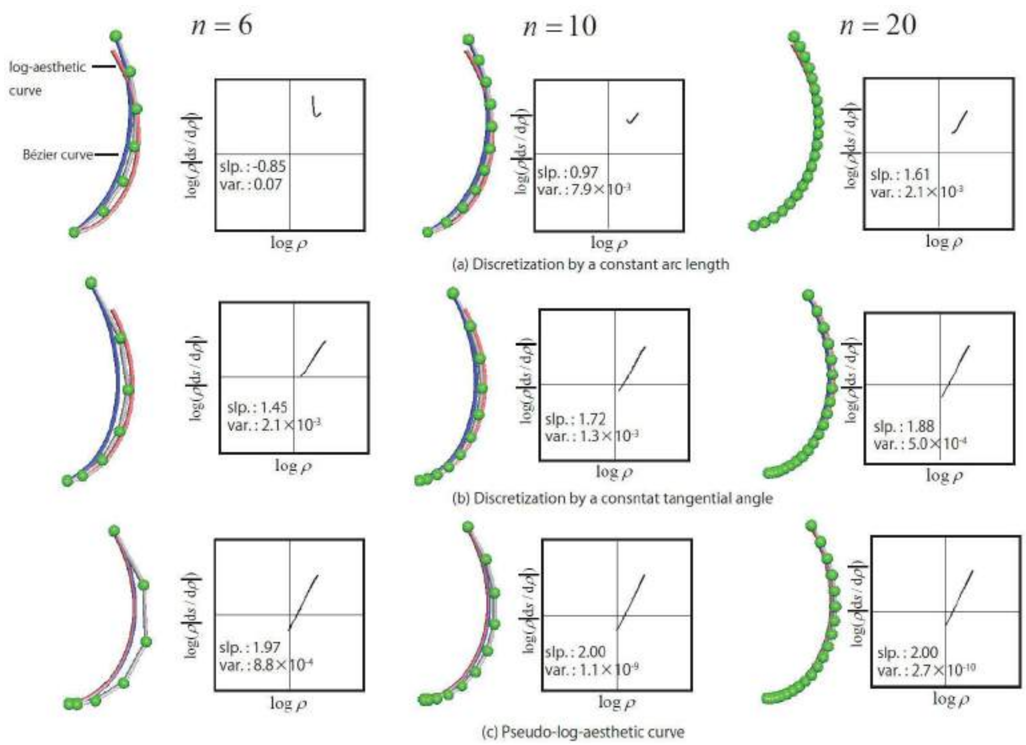


Fig. 7: Approximated log-aesthetic curves ($\alpha = 2$).

Concerning the Miura's method, the linearity of the LCG is better with the discretization of a constant arc length when $\alpha=-1$ than with a constant tangential angle. The linearity of the LCG is also better with a constant tangential angle when $\alpha=0,1,2$. In this section, when we refer to the discretization method, we mean the discretization with a constant arc length if $\alpha=-1$ or a constant arc length if $\alpha=0,1,2$ because they produce better linearity of LCGs. In cases of degrees 6 and 10, the slopes of LCGs of quasi-log-aesthetic curves are closer to the original α than the discretization method and the variance are smaller, except when $\alpha=0$. When $\alpha=0$, the discretization by a constant tangential angle produces better slopes of the LCGs and variances than quasi-log-aesthetic curves, but we can see that Bézier curve segment is not close to the original log-aesthetic curves shown in red. This can be considered that in the discretization by a constant tangential angle, a log-aesthetic curve with a similarity transformation (log-aesthetic curve with different Λ) is generated. The quasi-log-aesthetic curves can generate better linearity of LCGs than the discretization method when the degree is relatively low. When the degree is 20, the LCGs of quasi-log-aesthetic curves are not significantly better than the discretization method. This is because all of the curves generated by the three methods get closer to log-aesthetic curves when the degree gets higher.

In comparison with quasi-log-aesthetic curves in rational cubic Bézier form [10], the linearity of the LCGs of the rational cubic Bézier curves is always better than polynomial curves proposed in this paper. Especially, when log-aesthetic curve gets closer to a circular arc, quasi-log-aesthetic curves in polynomial Bézier form behave badly because polynomial curves cannot represent circular arcs. In such a situation, the curvature becomes not monotonically varying. However, log-aesthetic curves are meaningful for aesthetic shape design when they are not close to circular arcs. Thus quasi-log-aesthetic curves proposed are meaningful when approximate linearity of the LCG is required or polynomial Bézier curves are required.

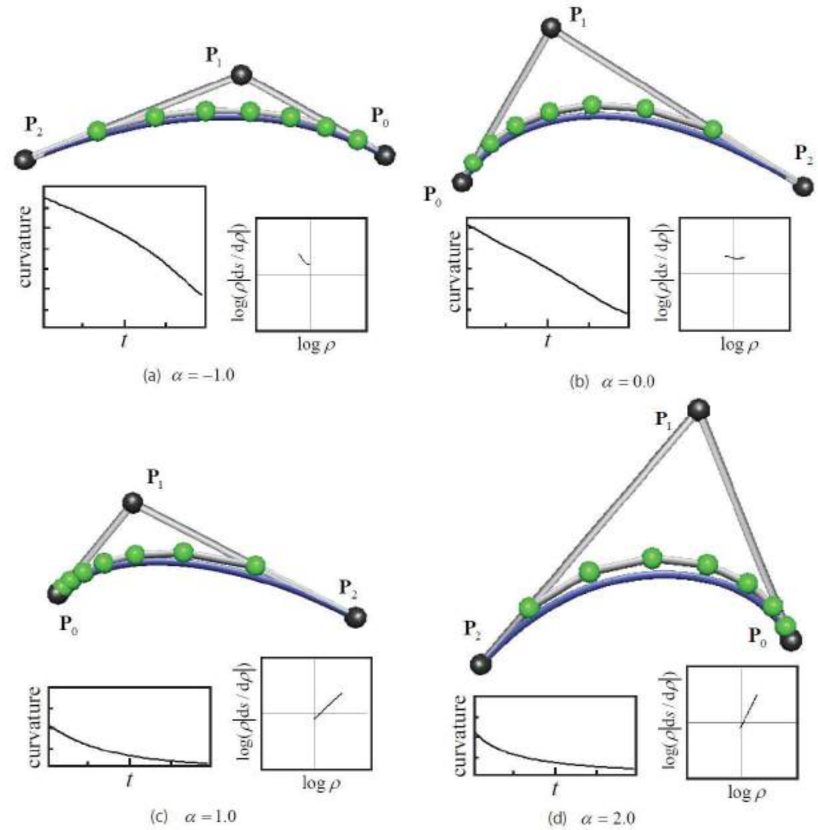


Fig. 8: G^1 Hermite interpolation.

Fig. 8 shows the results of G^1 Hermite interpolation, the curvature plots and the LCGs. Given $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$ and α, quasi-log-aesthetic curves in polynomial Bézier curve form of degree 8 are interactively generated. The curves are interactively generated by changing the three points and α. The linearity of the LCG gets worse if the curve includes nearby point of $\rho = 0$ or ∞ or if the curve is close to a circular arc. This is due to the limitation of the representation space of polynomial curves. Similarly as in log-aesthetic curves, the curve may not be found depending on the position $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$ and α. Making the degree higher, quasi-log-aesthetic curves get closer to log-aesthetic curves.

To inspect the geometric quality, we compare the swept surface of quasi-log-aesthetic curve($\alpha = 0$) with the swept surface of a curve with not monotonically varying curvature. Fig. 9 on the left shows the zebra mapping of the swept surface a curve with not monotonically varying curvature. The zebra lines are severely distorted and not aesthetically pleasing. Fig. 9 on the left shows the zebra mapping of the swept surface of quasi-log-aesthetic curve (polynomial Bézier curve with degree 8) with $\alpha = 0$ using G^1 Hermite interpolation algorithm described in Section 4. Zebra mappings of the swept surfaces of different α show similar results shown in Fig. 9 on the right, which are more aesthetically pleasing than Fig. 9 on the left. Thus, quasi-log-aesthetic curves have a potential to be used in

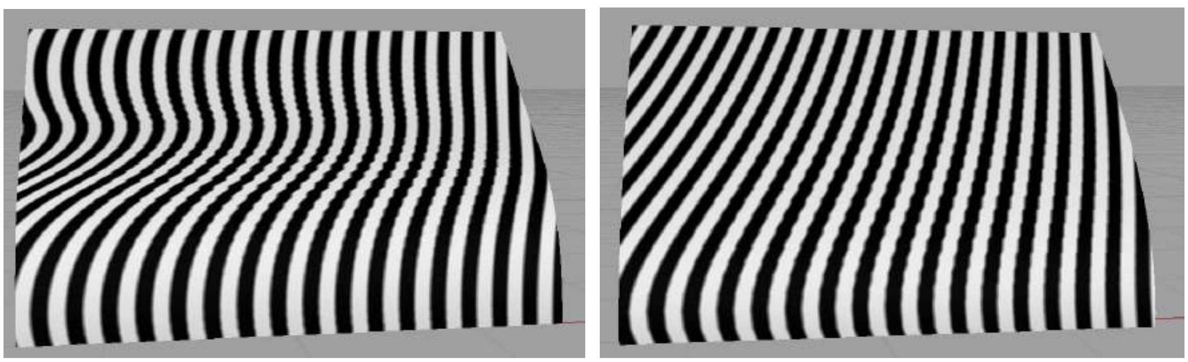


Fig. 9: Zebra mappings of the swept surface of a curve with not monotonically varying curvature (left) and the swept surface of quasi-log-aesthetic curve with $\alpha = 0$ (right).

6 CONCLUSIONS

This paper proposed quasi-log-aesthetic curves in polynomial Bézier form using Taylor polynomials. We have shown that quasi-log-aesthetic curves are better approximation than the curves generated by discretization [6]. Because of the limitation of polynomial curves, quasi-log-aesthetic curves are not good approximations when log-aesthetic curves are close to circular arcs. However, log-aesthetic curves are most useful when they are not close to circular arcs. We also proposed a method for G^1 Hermite Interpolation.

There are several directions for future research. One is to find a better method of approximation than our approach. Theoretically finding a way to switch between tangential angle equations and arc length equations depending on α and Λ may be possible. For this purpose, theoretical error analysis of the Taylor polynomials may be necessary. Theoretically finding the best θ_a is also important. Another direction is to approximate log-aesthetic space curves [11] in terms of freeform curves achieving good linearity of logarithmic and torsion curvature graphs. The idea of using Taylor polynomials cannot be used for log-aesthetic space curves, because they are drawn by solving the simultaneous differential equations.

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