Interactive Shape Control with Rational Cubic Splines

Zulfiqar Habib[1], Manabu Sakai[2] and Muhammad Sarfraz[3]

[1]Kagoshima University Japan, drzhabib@ybb.ne.jp
[2] Kagoshima University Japan, msakai@sci.kagoshima-u.ac.jp
[3]King Fahd University Saudi Arabia, sarfraz@kfupm.edu.sa

ABSTRACT

A rational cubic spline, with shape parameters, has been discussed with the view to its application in Computer Graphics. It incorporates both conic sections and parametric cubic curves as special cases. The parameters (weights), in the description of the spline curve can be used to modify the shape of the curve, locally and globally. The rational cubic spline attains parametric C^2 smoothness whereas the stitching of the conic segments preserves visually reasonable smoothness (C^1) at the neighboring knots. A very simple distance-based approximated derivative scheme is also presented to calculate control points. The curve scheme is interpolatory and can plot parabolic, hyperbolic, elliptic, and circular splines independently as well as bits and pieces of a rational cubic spline. We discuss difficult cases of elliptic arcs in space and introduce intermediate point interpolation scheme which can force the curve to pass through given point between any segment.

Keywords: Curves and Surfaces, Rational cubic spline, Conic, Shape preserving, Interpolation.

1. INTRODUCTION

A common problem, in Computer Graphics, is to design a curved outline by stitching small pieces of curves together. Piecewise rational cubic spline functions provide powerful tools for designing of curves, surfaces and some analytic primitives such as conic sections that are widely used in engineering design and various computer graphics applications. These applications may be representing some font outline [13], round corner in an object [3], or it may be a smooth fit to a given data [9]. Several segments of curves, to compose a desired curve outline, can have different mathematical descriptions. For example, a font "S" when designed, appears to have straight lines, conics, and cubics as essential parts of its outline. Single mathematical formulation for the precise definition of various types of geometry shapes is one of the major advantages of the rational cubic spline functions. This research is oriented towards similar direction and expected to achieve goal in terms of representing a piecewise parametric curve scheme which has all the features to produce a desired outline of shape.

In [2], C^1 rational cubic splines with exact derivatives for control points were used. We introduce a similar interpolant with a very simple distance-based approximated derivative scheme and achieved fine results. Our scheme is also more simple than area-based derivative scheme in [12]. Our research describes the parametric C^1 and C^2 rational cubic spline representation possessing a family of shape control parameters. This family of shape parameters has been utilized to produce straight line segments, conics, and cubics. The features of maintaining some reasonable amount of continuity (C^1) between conic and cubic arcs, estimated end derivatives, conic (circular, elliptical, parabolic, and hyperbolic) splines, circular arcs for given radius or center, elliptic arc in space and intermediate point interpolation are further achievements in this research. In [2], end derivatives are based on the assumption of the user, which is not convenient. Moreover, the conics were not discussed at all. We have estimated most suitable end derivatives for more pleasing results. In [10], cubic and conic segments are joined with G^1 continuity which is not reasonable for some practical applications. Intermediate point interpolation scheme and circular arcs, presented in [5], are not practical as the space curves and exact circular arcs are not possible. [11] offered intermediate point interpolation scheme with C^0 continuity at neighborhood points. [6] presented G^1 continuity in his recent research work on constrained guided curve scheme. He used rational quadratic function. We use rational cubic function and offer better continuity (C^1). In [4], rational quadratic spline is used for circular spline.

We have used very simple technique using rational cubic spline to achieve same circular spline.

We have used a very simple algorithm for any type of plane or space curve, parallel or non-parallel end tangents. Our scheme can generate exact circular and elliptical arcs. We have applied degree elevation techniques on rational quadratic spline as mentioned in [7]. NURBS (Non Uniform Rational B-Spline) representation of ellipse is given in [7]. We have improved this technique to handle any type of elliptic arcs even in space. In addition, the scheme has the following properties, which may lead to a more useful approach to curve and surface design in CAGD.

- The curve has C^2 continuity between the rational cubic arcs and C^1 continuity between cubic and conic arcs.
- Most suitable end derivatives are estimated.
- The scheme is local, i.e. shape control parameters will not significantly affect the adjacent parts of the design curve.
- A distance-based approximated derivative scheme is also used to compute control points. Tangent vectors vary continuously along the curve preserving C^1 continuity.
- Any part of the rational cubic spline can be made conic (with exact circle and ellipse) or straight line using the same interpolant.
- Intermediate point interpolation scheme has been introduced for use in guided curve.
- Our scheme can handle any kind of elliptic arc in space.
- All methods are suitable for space curves and hence can also be generalized to surfaces.

This paper has been organized in such a way that a parametric rational cubic spline scheme is considered in next section. Analysis of the designing curve has been made in section 3. In section 4, we present a scheme to calculate end derivatives (tangents). We discuss conditions for conics and straight line segments in section 5. This section also covers all types of circular and elliptical arcs in space and introduces a very powerful method for intermediate point interpolation. Examples are discussed in section 6. Finally, conclusion has been made in section 7.

2. THE RATIONAL CUBIC SPLINE

The cubic spline is the spline of the lowest degree with C^2 continuity. C^2 continuity meets the needs of most problems arising from engineering and mathematical physics. Rational cubic spline functions of lower degree are numerically simple, stable and fundamental of all rational space curves. Let $\boldsymbol{F}_i \in R^m$, $i=1,\ldots,n$, be a given set of points at the distinct knots $t_i \in R$, with unit interval spacing. Consider first degree parametric piecewise rational function for straight line segment between $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$:

$$\boldsymbol{L}(t) \equiv \boldsymbol{L}_i(t) = \frac{(1-s)\alpha_i \boldsymbol{F}_i + s\beta_i \boldsymbol{F}_{i+1}}{(1-s)\alpha_i + s\beta_i}, \qquad (1)$$

where $s=(t-t_i)/h_i$, $h_i=t_{i+1}-t_i$.

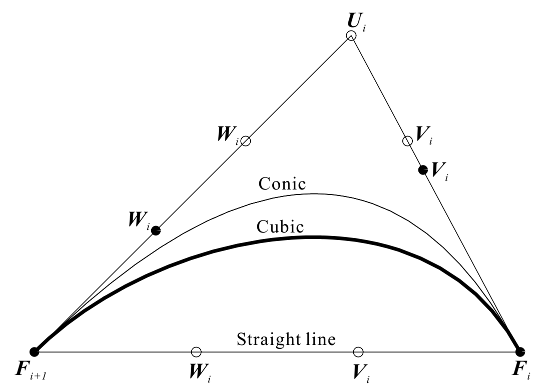


Fig. 1. Plot of $\boldsymbol{P}(t)$ with $\boldsymbol{V}_i$, $\boldsymbol{W}_i$ from (4) Straight line, (5) Conic and (6) Cubic.

We apply degree elevation formula ([1], p. 104) to get quadratic rational Bézier function:

$$\boldsymbol{Q}(t) \equiv \boldsymbol{Q}_i(t) = \frac{(1-s)^2 \alpha_i \boldsymbol{F}_i + s(1-s)\gamma_i \boldsymbol{U}_i + s^2 \beta_i \boldsymbol{F}_{i+1}}{(1-s)^2 \alpha_i + s(1-s)\gamma_i + s^2 \beta_i}, \qquad (2)$$

where $\boldsymbol{U}_i$ may be taken as the point of intersection of tangents at $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$ (see Fig. 1). Applying again degree elevation, we get rational cubic Bézier function:

$$\boldsymbol{P}(t) \equiv \boldsymbol{P}_i(t) = \frac{\boldsymbol{N}_1}{N_2}, \qquad (3)$$

where

$$\boldsymbol{N}_1 = (1-s)^3 \alpha_i \boldsymbol{F}_i + s(1-s)^2(\alpha_i+\gamma_i)\boldsymbol{V}_i + s^2(1-s)(\beta_i+\gamma_i)\boldsymbol{W}_i + s^3\beta_i \boldsymbol{F}_{i+1},$$

$$N_2 = (1-s)^2 \alpha_i + s(1-s)\gamma_i + s^2 \beta_i,$$

which is a straight line segment between $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$ with control points:

$$\boldsymbol{V}_i = \frac{2\alpha_i \boldsymbol{F}_i + \beta_i \boldsymbol{F}_{i+1}}{2\alpha_i + \beta_i}, \quad \boldsymbol{W}_i = \frac{\alpha_i \boldsymbol{F}_i + 2\beta_i \boldsymbol{F}_{i+1}}{\alpha_i + 2\beta_i}, \qquad (4)$$

and weight $\gamma_i = \alpha_i + \beta_i$. Similarly this function (3) is a conic curve between $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$ with following control points:

$$\boldsymbol{V}_i = \frac{\alpha_i \boldsymbol{F}_i + \gamma_i \boldsymbol{U}_i}{\alpha_i + \gamma_i}, \boldsymbol{W}_i = \frac{\beta_i \boldsymbol{F}_{i+1} + \gamma_i \boldsymbol{U}_i}{\beta_i + \gamma_i}, \tag{5}$$

Thus only one interpolant (3) is enough for straight line segment, conic arc and cubic arc. It is C^1 Hermite function for:

$$\begin{aligned} \boldsymbol{V}_i &= \boldsymbol{F}_i + \frac{\alpha_i}{\alpha_i + \gamma_i} \boldsymbol{D}_i, \\ \boldsymbol{W}_i &= \boldsymbol{F}_{i+1} - \frac{\beta_i}{\beta_i + \gamma_i} \boldsymbol{D}_{i+1}. \end{aligned} \tag{6}$$

This can be achieved by imposing the Hermite interpolation conditions:

$$\boldsymbol{P}(t_i) = \boldsymbol{F}_i \text{ and } \boldsymbol{P}^{(1)}(t_i) = \boldsymbol{D}_i, \ \forall_i. \tag{7}$$

3. DESIGN CURVE ANALYSIS

The parameters α_i, β_i and γ_i are mainly meant to be used freely to control the shape of the curve. At the same time, for the convenient of the designer, it is also required that the ideal geometric properties of the curve are not lost. The geometric properties, like variation diminishing, convex hull, and positivity, are the ones which need to be presented in the description of the design curve.

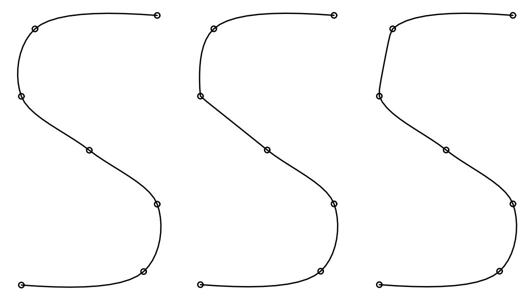
- For the constraints, $\alpha_i>0$, $\beta_i>0$ and $\gamma_i>-\alpha_i$, $-\beta_i$, $\forall i$, it is very obvious that the rational cubic is characterized as of Bernstein Bézier form. The case for default values of shape parameters $\alpha_i=1=\beta_i$ and $\gamma_i=2$ is that of cubic Hermite interpolation.
- Thus following the Bernstein Bézier theory, the piece of curve $\boldsymbol{P}_i(t)$ lies in the convex hull of $\boldsymbol{F}_i$, $\boldsymbol{V}_i$, $\boldsymbol{W}_i$, $\boldsymbol{F}_{i+1}$.
- It also follows the variation diminishing property within the convex hull. That is any straight line crossing the control polygon of $\boldsymbol{F}_i$, $\boldsymbol{V}_i$, $\boldsymbol{W}_i$, $\boldsymbol{F}_{i+1}$ does not cross the curve more than its control polygon.

3.1 Point and Interval Tension

The following `tension' properties of the rational Hermite form are now immediately apparent from (3) and (6), (see Fig. 2).

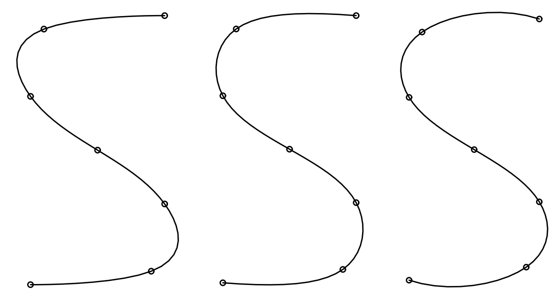
3.1.1 Point Tension

Accentuated point tension can be achieved by considering $\alpha_{i-1}=\beta_i\to 0$. The point tension property holds from both right and left of t_i, where the spline interpolant becomes C^0. This case thus allows the introduction of a tangent discontinuity at t_i.



(a) Default: $\alpha_i=1=\beta_{i-1}$, $\gamma_i=2$ (b) Point tension: $\alpha_3=0.2=\beta_2$ (c) Interval tension: $\gamma_2=20$

Fig. 2. Demonstration of shape parameters using distance-based derivatives (C^1 continuity).



(a) Supposed (b) Distance based (c) Reasonable

Fig. 3. Demonstration of end derivatives using exact derivatives (C^2 continuity).

$$\lim_{\alpha_i \to 0} \boldsymbol{V}_i = \boldsymbol{F}_i \text{ and } \lim_{\alpha_i \to 0} \boldsymbol{P}_i(t) = \frac{(1-s)^2 \gamma_i \boldsymbol{F}_i + s(1-s)(\gamma_i + \beta_i)\boldsymbol{W}_i + s^2 \beta_i \boldsymbol{F}_{i+1}}{(1-s)\gamma_i + s\beta_i},$$

$$\lim_{\beta_i \to 0} \boldsymbol{W}_i = \boldsymbol{F}_{i+1} \text{ and } \lim_{\beta_i \to 0} \boldsymbol{P}_i(t) = \frac{(1-s)^2 \alpha_i \boldsymbol{F}_i + s(1-s)(\gamma_i + \alpha_i)\boldsymbol{V}_i + s^2 \gamma_i \boldsymbol{F}_{i+1}}{(1-s)\alpha_i + s\gamma_i}.$$

See Fig. 2(b)., where point tension is increased at 3rd point by decreasing the values of α_3 and β_2.

3.1.2 Interval Tension

The interval shape property is obvious from the following limit behavior. That is, the increase in the shape parameter γ_i in any interval i tightens the curve towards the line segment joined by the control points and the resulting rational spline interpolant is C^1 at t_i and t_{i+1}.

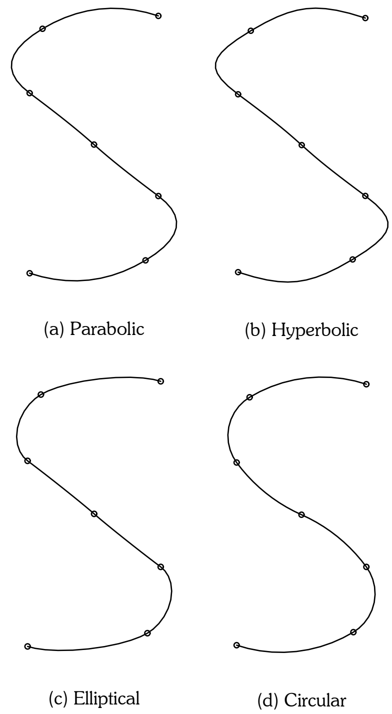


Fig. 4. Conic spline.

See Fig. 2(c)., where interval tension is increased for 2nd segment by increasing the value of γ_2.

$$\lim_{\gamma_i \to \infty} \boldsymbol{V}_i = \boldsymbol{F}_i, \ \lim_{\gamma_i \to \infty} \boldsymbol{W}_i = \boldsymbol{F}_{i+1} \ \text{ and}$$
$$\lim_{\gamma_i \to \infty} \boldsymbol{P}_i(t) = (1-s)\boldsymbol{F}_i + s\boldsymbol{F}_{i+1}. \tag{8}$$

4. TANGENT VECTORS

There are different choices of the tangent vectors $\boldsymbol{D}_i$ at $\boldsymbol{F}_i$, which can be opted for practical implementation for the computation of a curve with specific amount of smoothness. For curve methods, some reasonable tangent approximation method can be used. The distance-based approximations are found reasonably good as far as pleasing smoothness is concerned. For a higher continuity, for example C^2 rational cubic spline, complicated constraints are required to be fit. Readers are referred to [10] for detail.

4.1 Estimation of End Tangent Vectors

Tangent vectors for end segments are usually supposed but unfortunately these are not visually pleasing always. To make the end segments more appropriate, a compatible choice for the curve scheme of this paper, is presented here. For tangent at first point, let θ_1 be the angle between $\boldsymbol{F}_3$-$\boldsymbol{F}_1$ and $\boldsymbol{F}_2$-$\boldsymbol{F}_1$. Let $\boldsymbol{T}_1$ be the rotation of $\boldsymbol{F}_2$ around $\boldsymbol{F}_1$ by an angle θ_1 on the plane passing through $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$. Now we derive tangent vector $\boldsymbol{D}_1$ at first point as follows:

$$\mu_1 = \frac{(\boldsymbol{F}_2 - \boldsymbol{F}_1)^2}{2(\boldsymbol{F}_2 - \boldsymbol{F}_1)\cdot \boldsymbol{T}_1}, \ \boldsymbol{U}_1 = \boldsymbol{F}_1 + \mu_1 \boldsymbol{T}_1,$$
$$\boldsymbol{V}_1 = \frac{\boldsymbol{F}_1 + 2\boldsymbol{U}_1}{3}, \ \boldsymbol{D}_1 = 3(\boldsymbol{V}_1 - \boldsymbol{F}_1), \tag{9}$$

where μ_1 is determined by the condition:

$$|\boldsymbol{U}_1 - \boldsymbol{F}_1| = |\boldsymbol{U}_1 - \boldsymbol{F}_2|. \tag{10}$$

Similarly, for tangent vector $\boldsymbol{D}_n$ at last point, let θ_n be the angle between $\boldsymbol{F}_{n-2}$-$\boldsymbol{F}_n$ and $\boldsymbol{F}_{n-1}$-$\boldsymbol{F}_n$. Let $\boldsymbol{T}_n$ be the rotation of $\boldsymbol{F}_{n-1}$ around $\boldsymbol{F}_n$ by an angle θ_n on the plane passing through $\boldsymbol{F}_n$, $\boldsymbol{F}_{n-1}$ and $\boldsymbol{F}_{n-2}$. Then

$$\mu_{n-1} = \frac{(\boldsymbol{F}_{n-1} - \boldsymbol{F}_n)^2}{2(\boldsymbol{F}_{n-1} - \boldsymbol{F}_n)\cdot T_n}, \ \boldsymbol{U}_{n-1} = \boldsymbol{F}_n + \mu_{n-1}\boldsymbol{T}_n,$$
$$\boldsymbol{W}_{n-1} = \frac{\boldsymbol{F}_n + 2\boldsymbol{U}_{n-1}}{3}, \ \boldsymbol{D}_n = 3(\boldsymbol{F}_n - \boldsymbol{W}_{n-1}),$$

where μ_{n-1} is determined by the condition:

$$|\boldsymbol{U}_{n-1} - \boldsymbol{F}_n| = |\boldsymbol{U}_{n-1} - \boldsymbol{F}_{n-1}|. \tag{11}$$

Visual difference between different types of end tangent vectors has been demonstrated in Fig. 3.

5. CONIC SPLINE AND STRAIGHT LINE

Conic and straight line are the most important parts in designing which can be achieved through rational cubic interpolant (3), so that we can use the same interpolant for all types of curves. As mentioned before, $\boldsymbol{U}_i$ is the point of intersection of tangents at $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$. In case the tangents are parallel, $\boldsymbol{U}_i$ can be taken as the point where the arc is desired to be divided into two pieces, for example, it may be the inflection or the middle point, etc. For conic section properties and choice of shape parameters, various conics are recovered depending upon the nature of weights [8]. Also readers are referred to ([7], p. 291-295) and ([1], p. 73-96). According to [7], the conic shape factor:

$$k = \frac{\alpha_i \beta_i}{\gamma_i^2} \tag{12}$$

determines the conic; if the three weights are changed in such a way that k is not changed. Thus any two weights can be chosen arbitrarily; the conic is then determined by the third weight. It is customary to choose $\alpha_i = 1 = \beta_i$. The C^1 conic spline is:

- Parabolic if γ_i=2 (Fig. 4(a)).
- Hyperbolic if γ_i>2 (Fig. 4(b)).
- Elliptic if -1<γ_i<2 (Fig. 4(c)).

5.1 Conic Arc in Cubic Spline

Rational cubic interpolant (3) can easily adjust conic segment in cubic spline. Cubic segments are already joined by C^2 continuity but we also need some smoothness between conic and cubic segments. C^1 continuity is enough for visually pleasing results. Let i-th segment between $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$ is conic. If $i>1$, then for C^1 continuity at $\boldsymbol{F}_i$, we impose the constraints $\boldsymbol{P}^{(1)}(t_i^-) = \boldsymbol{P}^{(1)}(t_i^+)$ to find

$$\boldsymbol{W}_{i-1} = \frac{(2+\gamma_{i-1}+\gamma_i)\boldsymbol{F}_i - (1+\gamma_i)\boldsymbol{V}_i}{1+\gamma_{i-1}}. \tag{13}$$

If $i<n$, then for C^1 continuity at $\boldsymbol{F}_{i+1}$, we impose the constraints $\boldsymbol{P}^{(1)}(t_{i+1}^-) = \boldsymbol{P}^{(1)}(t_{i+1}^+)$ to find

$$\boldsymbol{V}_{i+1} = \frac{(2+\gamma_i+\gamma_{i+1})\boldsymbol{F}_{i+1} - (1+\gamma_i)\boldsymbol{W}_i}{1+\gamma_{i+1}}. \tag{14}$$

5.2 Circular Spline

For G^1 circular spline, see Fig. 5., and consider:

$$\gamma_i = 2\cos\phi, \tag{15}$$

where ϕ is the angle between $\boldsymbol{F}_{i+1}$-$\boldsymbol{F}_i$ and $\boldsymbol{U}_i$- $\boldsymbol{F}_i$. Let $\boldsymbol{T}_i$ be the unit vector along $\boldsymbol{D}_i$ and $\boldsymbol{U}_i$, the point of intersection of tangent vectors at $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$ is:

$$\boldsymbol{U}_i = \boldsymbol{F}_i + \mu_i \boldsymbol{T}_i, \tag{16}$$

where μ_i is determined by the condition:

$$|\boldsymbol{U}_i - \boldsymbol{F}_i| = |\boldsymbol{U}_i - \boldsymbol{F}_{i+1}|, \tag{17}$$

from which we have:

$$\mu_i = \frac{(\boldsymbol{F}_{i+1} - \boldsymbol{F}_i)^2}{2(\boldsymbol{F}_{i+1} - \boldsymbol{F}_i)\cdot \boldsymbol{T}_i}. \tag{18}$$

Circular spline is given in Fig. 4(a). Fig. 4(a)., shows three point exact circle.

5.3 Circular Arc

This section is devoted for the construction of circular arc. The cases, for a given radius and given center, are independently discussed.

5.3.1 Circular Arc For Given Radius

Let r be the given radius of the circular arc such that $r>(|\boldsymbol{F}_{i+1}$-$\boldsymbol{F}_i|)/2$. Then, the center $\boldsymbol{M}$ can lie anywhere on the circle centered at $\boldsymbol{N}=(\boldsymbol{F}_i+\boldsymbol{F}_{i+1})/2$ and having radius b as follows:

$$b = \sqrt{r^2 - \frac{|\boldsymbol{F}_{i+1} - \boldsymbol{F}_i|^2}{4}}. \tag{19}$$

It will be preferred that $\boldsymbol{M}$ should lie on the plane passing through $\boldsymbol{F}_i$, $\boldsymbol{F}_{i+1}$ and $\acute{\boldsymbol{U}}_i$, where $\acute{\boldsymbol{U}}_i$ is the intersection of $\boldsymbol{F}_i$-$\boldsymbol{D}_i$ and $\boldsymbol{F}_{i+1}$-$\boldsymbol{D}_{i+1}$. Therefore circular arc should lie on the side of $\acute{\boldsymbol{U}}_i$. Let $\boldsymbol{e}_1$ be the rotation of $\boldsymbol{F}_{i+1}$ around $\boldsymbol{N}$ by an angle θ on the plane passing through $\boldsymbol{F}_i$, $\boldsymbol{F}_{i+1}$ and $\acute{\boldsymbol{U}}_i$, where $\theta=\pi/2$ for anti-clockwise rotation and $\theta=-\pi/2$ for clockwise rotation.

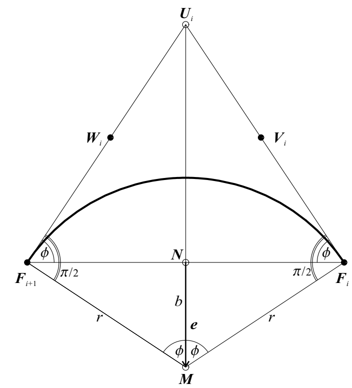


Fig. 5. Bézier points of Circular Arc.

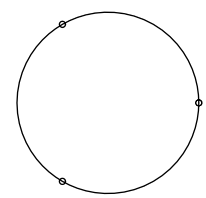


Fig. 6. A three-point circle given in rational cubic Bézier form.

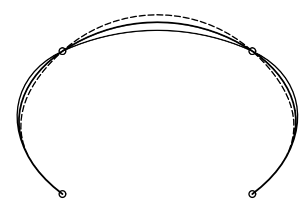


Fig. 7. Rational cubic spline with mid interval as circular arc piece for radius $r=15$ (dashed), 18 (bold), 24 (normal).

Now, $\boldsymbol{e}=(\boldsymbol{e}_1$-$\boldsymbol{N})/|\boldsymbol{e}_1$-$\boldsymbol{N}|$ is a unit vector passing through $\boldsymbol{N}$ and perpendicular to $\boldsymbol{F}_{i+1}$-$\boldsymbol{F}_i$. Then, $\boldsymbol{M}=\boldsymbol{N}+b\boldsymbol{e}$ will be the center of our required circular arc. Let $\phi=\angle \boldsymbol{F}_i\boldsymbol{M}\boldsymbol{N}$. Replace ϕ with -ϕ if circular arc rotation is anti-clockwise. Next, we find γ_i from (15). Let $\boldsymbol{T}'$ be the rotation of $\boldsymbol{F}_{i+1}$ around $\boldsymbol{F}_i$ through angle ϕ on the plane passing through $\boldsymbol{F}_i$, $\boldsymbol{F}_{i+1}$ and $\acute{\boldsymbol{U}}_i$ from which we have $\boldsymbol{T}_i=(\boldsymbol{T}'$-$\boldsymbol{F}_i)/|\boldsymbol{T}'$-$\boldsymbol{F}_i|$, a unit tangent vector at $\boldsymbol{F}_i$. Now use (16) to find $\boldsymbol{U}_i$, (5) to find control points $\boldsymbol{V}_i$ and $\boldsymbol{W}_i$, (13) for C^1 continuity at $\boldsymbol{F}_i$, (14) for C^1 continuity at $\boldsymbol{F}_{i+1}$ and finally use rational cubic interpolant (3) for required circular arc. In this scheme the radius r can be used as a shape control parameter demonstrated in Fig. 7.

5.3.2 Circular Arc For Given Center

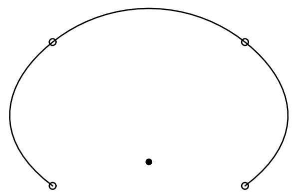


Fig. 8. Rational cubic spline with mid interval as circular arc piece for given center.

Let $\boldsymbol{M}$ be the given center of the circular arc such that $|\boldsymbol{F}_{i+1}- \boldsymbol{M}|=|\boldsymbol{F}_i-\boldsymbol{M}|$. Let $\boldsymbol{M}'$ be the rotation of $\boldsymbol{M}$ around $\boldsymbol{F}_i$ by an angle θ on the plane passing through $\boldsymbol{F}_i$, $\boldsymbol{F}_{i+1}$ and $\boldsymbol{M}$, where $\theta=\pi/2$ for clockwise rotation and $\theta=-\pi/2$ for anti-clockwise rotation. $\boldsymbol{T}_i=(\boldsymbol{M}'- \boldsymbol{F}_i)/|\boldsymbol{M}'-\boldsymbol{F}_i|$ is a unit tangent vector at $\boldsymbol{F}_i$. Let ϕ be the angle between $\boldsymbol{F}_{i+1}$-$\boldsymbol{F}_i$ and $\boldsymbol{T}_i$. Now use (15) to find γ_i, (16) to find $\boldsymbol{U}_i$, (5) to find control points $\boldsymbol{V}_i$ and $\boldsymbol{W}_i$, (13) for C^1 continuity at $\boldsymbol{F}_i$, (14) for C^1 continuity at $\boldsymbol{F}_{i+1}$ and finally use rational cubic interpolant (3) for required circular arc. Fig. 8., shows the plot of *C*-type rational cubic spline with mid segment as circular arc. The center of this circular arc is shown as small disk where as given data is shown as small circles.

5.4 Elliptic Arc

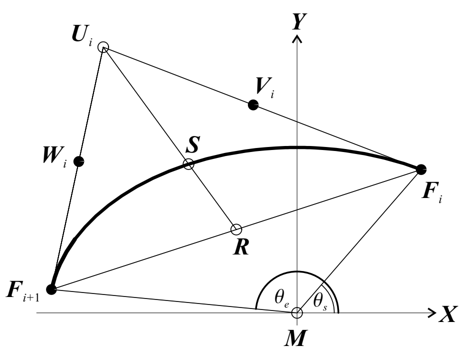


Fig. 9. Bézier points of Elliptic Arc.

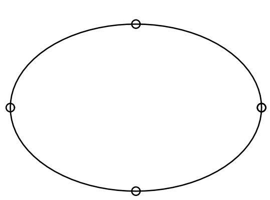


Fig. 10. A four-point ellipse given in rational cubic Bézier form.

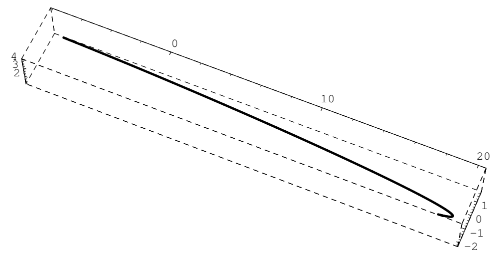


Fig. 11. An elliptic arc in space.

This section is devoted for the construction of elliptic arc in three dimension. Very complicated cases have also been treated, e.g., when major axis becomes too much larger than minor axis and required elliptic arc consists of highest curvature part of the ellipse.

Given start point $\boldsymbol{F}_i$, end point $\boldsymbol{F}_{i+1}$, center $\boldsymbol{M}$, unit vector along major axis $\boldsymbol{X}$, unit vector along minor axis $\boldsymbol{Y}$, semi major axis *a* and semi minor axis *b* (see Fig. 9). $\boldsymbol{XMY}$ is a local coordinate system in space. Let $\theta_s=\angle\boldsymbol{XMF}_i$ and $\theta_e=\angle\boldsymbol{XMF}_{i+1}$. If necessary, use Newton Raphson method to compute θ_s and θ_e. If $\theta_s>\theta_e$, replace θ_s with θ_s-2π. $\boldsymbol{S}(=\boldsymbol{M}+\text{a} \cos\theta\, \boldsymbol{X}+\text{b} \sin\theta\, \boldsymbol{Y}\})$ is a point on elliptic arc, where $\theta=(\theta_s+\theta_e)/2$. Let $\boldsymbol{U}_i$ be the point of intersection of tangents $\boldsymbol{T}_0(=-\text{a} \sin\theta_s \boldsymbol{X}+\text{b} \cos\theta_s \boldsymbol{Y})$ and $\boldsymbol{T}_1(=-\text{a} \sin\theta_e \boldsymbol{X}+\text{b} \cos\theta_e \boldsymbol{Y})$ at $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$ respectively. Let $\boldsymbol{R}$ be the point of intersection of $\boldsymbol{S}$-$\boldsymbol{U}_i$ and $\boldsymbol{F}_{i+1}$-$\boldsymbol{F}_i$. The quadratic rational Bézier arc (3) can be written in the form:

$$\boldsymbol{Q}(u)=\frac{(1-u)^2\boldsymbol{F}_i+u(1-u)\gamma_i\boldsymbol{U}_i+u^2\boldsymbol{F}_{i+1}}{(1-u)^2+u(1-u)\gamma_i+u^2}. \quad (20)$$

Now the line $\boldsymbol{L}(u)=[\boldsymbol{F}_i, \boldsymbol{F}_{i+1}]$ is obtained by taking $\gamma_i=0$. Therefore

$$\boldsymbol{L}(u)=\frac{(1-u)^2\boldsymbol{F}_i+u^2\boldsymbol{F}_{i+1}}{(1-u)^2+u^2}, \quad (21)$$

which is convex combination of $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$ and

$$\frac{|\boldsymbol{R}-\boldsymbol{F}_i|}{|\boldsymbol{R}-\boldsymbol{F}_{i+1}|}=\frac{u^2}{1-u^2}. \quad (22)$$

Then $u=c/(1+c)$, where

$$c=\sqrt{\frac{|R-F_i|}{|R-F_{i+1}|}}. \quad (23)$$

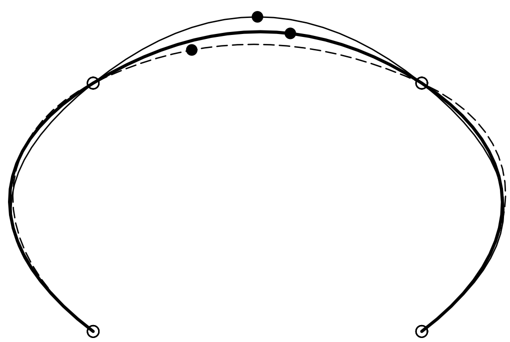
Therefore, $\boldsymbol{Q}(u)=\boldsymbol{S}$ and from (20), we can easily find

$$\gamma_i=\frac{1}{u(1-u)|\boldsymbol{U}_i-\boldsymbol{S}|}\times\left\{(1-u)^2(\boldsymbol{S}-\boldsymbol{F}_i)+u^2(\boldsymbol{S}-\boldsymbol{F}_{i+1})\right\}\cdot(\boldsymbol{U}_i-\boldsymbol{S}). \quad (24)$$

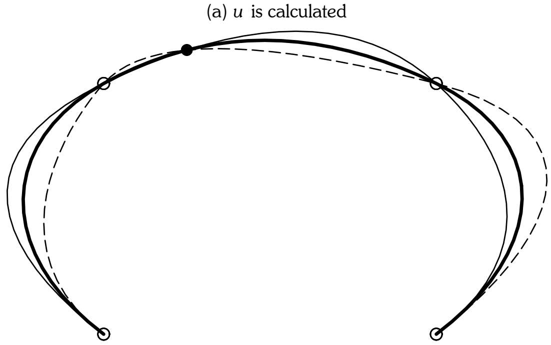
Now use (5) to find control points $\boldsymbol{V}_i$ and $\boldsymbol{W}_i$ and rational cubic interpolant (3) for required elliptic arc. Four point ellipse given in rational cubic Bézier form. Fig. 11., is showing an elliptic arc in space from following given information:

a=20, b=1, $\boldsymbol{M}$=(0,0,2),
$\boldsymbol{F}_i$=(18.967819, -2.184863, 3.943775),
$\boldsymbol{F}_{i+1}$=(-7.476452, 1.674027, 1.144135),
$\boldsymbol{X}$=(0.990033, -0.099335, 2.099833),
$\boldsymbol{Y}$=(0.109252,0.989038, 1.900665).

5.5 Intermediate Point Interpolation



(a) u is calculated



(b) u=0.2 (normal), 0.4 (dashed), calculated (bold)

Fig. 12. Intermediate point interpolation.

We need to insert point $\boldsymbol{C}$ between $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$ while preserving some reasonable continuity (C^1) at $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$. For $\alpha_i = 1 = \beta_i$, consider:

$$\boldsymbol{U}_i = \frac{1}{u(1-u)\gamma_i}\Big[\big\{(1-u)^2 + u(1-u)\gamma_i + u^2\big\}\boldsymbol{C} - (1-u)^2\boldsymbol{F}_i - u^2\boldsymbol{F}_{i+1}\Big],$$

where

$$u = \frac{|\boldsymbol{F}_i - \boldsymbol{C}|}{|\boldsymbol{F}_i - \boldsymbol{C}| + |\boldsymbol{F}_{i+1} - \boldsymbol{C}|}. \tag{25}$$

Next, use (5) to find control points $\boldsymbol{V}_i$ and $\boldsymbol{W}_i$, (13) for C^1 continuity at $\boldsymbol{F}_i$, (14) for C^1 continuity at $\boldsymbol{F}_{i+1}$ and finally use rational cubic interpolant (3) for required result. Fig. 12(a)., is showing an intermediate point interpolation in middle segment where the curve is forced to pass through different small disks.

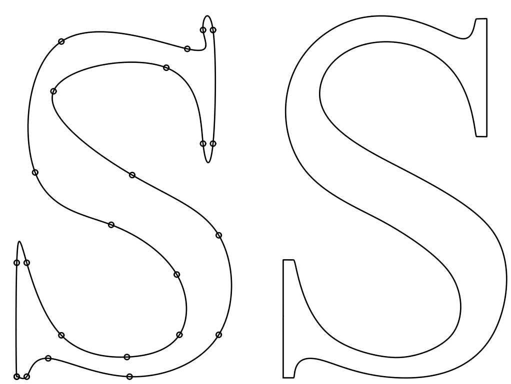
The parameter u can also be used as shape control parameter within the range $0<u<1$. For different values of u, we can construct a family of curves interpolating $\boldsymbol{C}$ (small disk) as shown in Fig. 12(b).

5.6 Straight Line Segment

For straight line segment using rational cubic interpolant (3), we have following four different methods:

1. Consider $\gamma_i = 0$.
2. Replace $\boldsymbol{U}_i$ with $\boldsymbol{F}_i$ or $\boldsymbol{F}_{i+1}$ and then use (5) to find control points $\boldsymbol{V}_i$ and $\boldsymbol{W}_i$.
3. Use intermediate point interpolation scheme by inserting point $\boldsymbol{C}$ on line joining $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$.
4. Consider $\gamma_i = \alpha_i + \beta_i$, then find control points $\boldsymbol{V}_i$ and $\boldsymbol{W}_i$ from (4).

6. EXAMPLES



(a) Default

(b) With interval and Point tension control

Fig. 13. Times new roman font "S" with rational cubic spline interpolation.

Data taken from times new roman font "S" has been interpolated by default rational cubic spline in Fig. 13(a). It is not as desired. Point and interval tension parameters are changed to achieve visually pleasing shape for font "S" in Fig. 13(b).

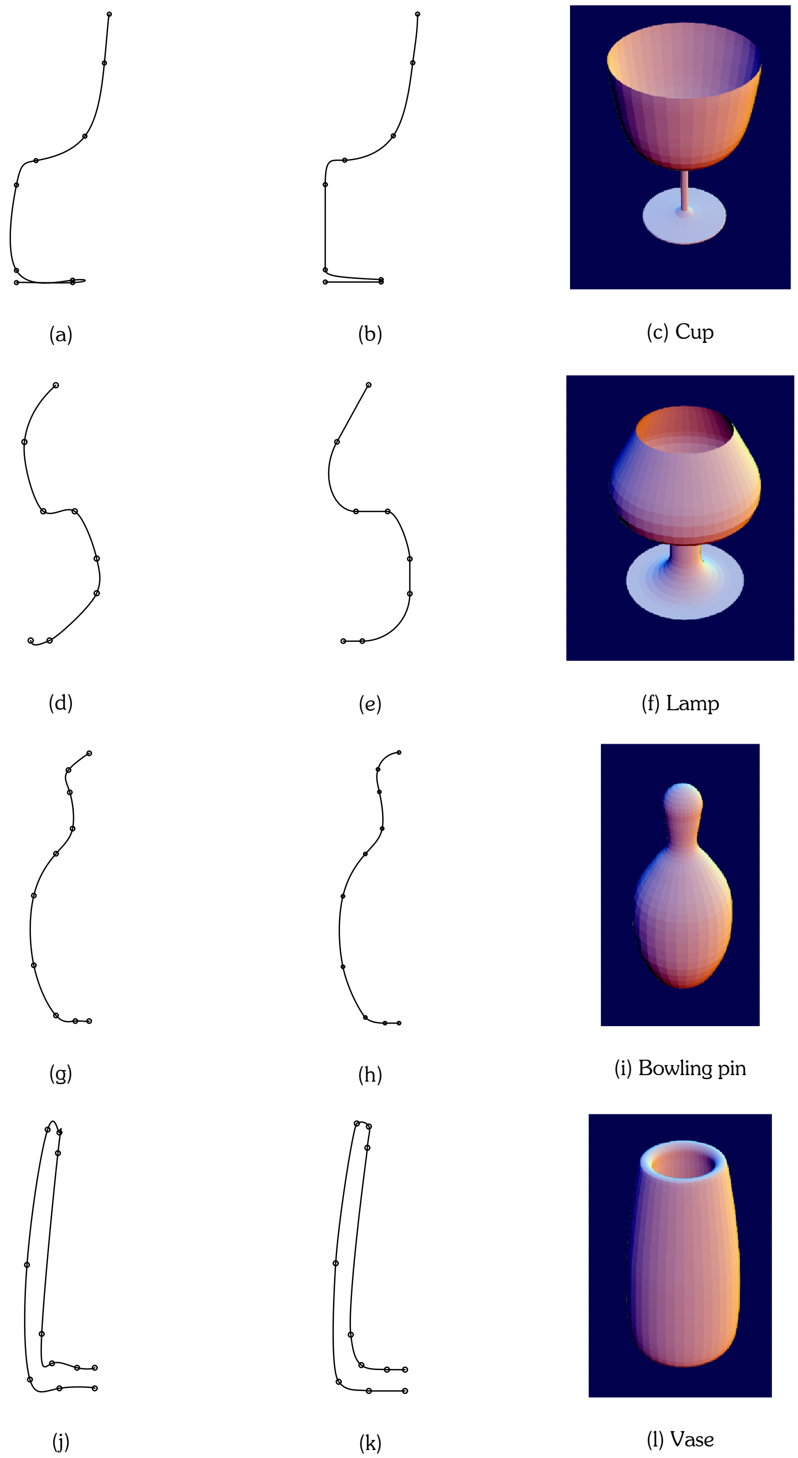


Fig. 14. Rational cubic spline (From left to right: Default, Well shaped, Shaded rendition of well shaped outlines).

Fig. 14., illustrates the design of a rational cubic spline used for a surface of revolution that represents a cup, lamp, bowling pin and vase. Figures 14(a)., 14(d)., 14(g)., and 14(j)., are default shapes with exact derivatives and use default values of shape parameters, i.e. $\alpha_i=1=\beta_i$ and $\gamma_i=2$. Figures 14(b)., 14(e)., and 14(h)., are also with exact derivatives whereas Fig. 14(k)., is plotted with distance-based approximated derivatives. To make these figures well shaped and pleasing, we used shape control parameters and inserted some conic or straight line segments connected by C^1 continuity with neighborhood cubic segments. Detail about these figures is as follows:

Fig. 14(b): from bottom, $\gamma_1=100$, $\gamma_3=10$ and $\gamma_4=100$. Second segment is a circular arc. All other segments are cubic connected by C^2 continuity and use default values of shape parameters.

Fig. 14(e): from bottom, $\gamma_1=100$, $\gamma_3=100$, $\gamma_5=100$, $\gamma_6=0.1$ and $\gamma_7=100$. Second segment is a circular arc with radius 15. All other segments are cubic connected by C^2 continuity and use default values of shape parameters.

Fig. 14(h): From top, first segment is a circular arc with radius 8, second last is a conic and last is a straight line. All other segments are cubic connected by C^2 continuity and use default values of shape parameters.

Fig. 14(k): from bottom, $\gamma_1=200$, $\gamma_7=0.01$, $\gamma_{n-1}=100$. Fifth segment is a circular arc. All other segments are cubic connected by C^1 continuity and use default values of shape parameters.

7. CONCLUSION

We have described an interval controlled rational cubic interpolation scheme. The scheme offers a number of possible ways in which the shape of the corresponding curves may be altered by the users. It is therefore felt that such a scheme could be a useful addition to an interactive design package, with the user having enough control over the curve segments. The provision of the shape parameters, in the description of the piecewise rational functions, provides freedom to modify the shape in desirous regions in a stable manner. The rational spline scheme is meant for parametric curves and is capable of designing plane as well as space curves. It is an interpolatory rational spline scheme enjoying all the ideal geometric properties. It has features to produce all types of conic curves in such a way that the whole design curve may be produced as a circular, elliptic, parabolic, or a hyperbolic spline curve. In addition, the desired conic pieces may also be fitted within the rational cubic spline. Overall smoothness of the rational cubic spline is C^2 whereas the conics are stitched with C^1 continuity. The curve scheme is extendable to surfaces.

8. ACKNOWLEDGEMENT

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