Geometric Design of Developable Composite Bézier Surfaces

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ABSTRACT

This paper studies geometric design of developable surfaces that consist of consecutive Bézier patches. It is shown that the number of degrees of freedom (DOF) for the surface design is independent of the degree of the surface. With a first boundary curve freely specified, (2m+3), (m+4), and five DOF's are available for a second boundary curve of a developable surface containing m patches, when the surface is G^0, G^1, and G^2, respectively. There remain five and (7–2m) DOF's for C^1 and C^2 continuity. Four and three DOF's are left for the patch design when the end ruling vanishes on one and both sides. Design examples are presented that fully utilize the corresponding DOF's subject to various continuity conditions. This work provides the foundation for systematic implementation of a CAGD system for developable composite Bézier surfaces.

Keywords: Developable surfaces, Bézier surfaces, triangular Bézier patches, CAGD

1. INTRODUCTION

Developable surfaces are widely used in design and manufacturing of materials that do not stretch or tear. Applications include modeling of ship hulls, apparel, ducts, automobile, and aircraft components [1]. Products are first designed using developable surfaces in 3D space; then they are flattened and become a 'pattern' in a plane. The fabrication process starts with cutting materials (such as sheet metal, paper, leather, or plywood) according to the pattern. The planar patterned materials are simply bent or un-rolled back to the original 3D shapes.

A ruled surface is generated by sweeping a straight line through 3D space [2]. This straight line is referred to as a ruling, or generator of the surface. Developable surfaces are a subset of ruled surfaces, which has a constant tangent plane at all points along any ruling. Mathematically a developable surface is defined as the envelope of a single family of planes. It is also known as a single curved surface, as one of its principal curvatures is null [3].

There are two main approaches to constructing developable surfaces. Aumann [4] proposed the condition under which a developable Bézier patch can be constructed with two boundary curves. They are restricted to lie in parallel planes and their projection in the x-y plane to be a rectangle. Lang and Röschel [5] obtained necessary conditions for the control nets and the weights of a rational Bézier surface to become developable. However, the result leads to a complex system of coupled equations that make it difficult to design developable surfaces with their method. Frey and Bindschadler [6] extended the results obtained by Aumann into developable Bézier strip patches that allow their projections to be general trapezoids in the x-y plane. Chalfant and Maekawa [7] studied design of a developable B-spline surface by joining m developable Bézier patches along their end rulings with C^2 continuity, but their method restricts the two boundary curves to lie in parallel planes. Their later work [8] developed a method for design of developable surfaces with general 3D boundary curves. Optimization techniques are employed to compute the remaining control points after the user has designated the first curve and the end points of the other, but the solved surface may not always be precisely developable. The second approach [9,10] constructs a developable surface in terms of plane geometry using the concept of duality between points and planes in 3D projective space. This method provides a compact representation for developable surfaces in the dual form, but it is difficult to apply the results to computer-aided geometric design of 3D shapes.

All previous studies impose the original constraint – constant tangent plane at all points along any given ruling, on the control net of a surface to guarantee its developability. By doing so, they failed to infer important properties of a developable control polygon that facilitate the solution process of the surface design. As a result, to determine the constrained control points involves the solution of highly coupled non-linear systems of equations. More importantly, those studies lost insights into the

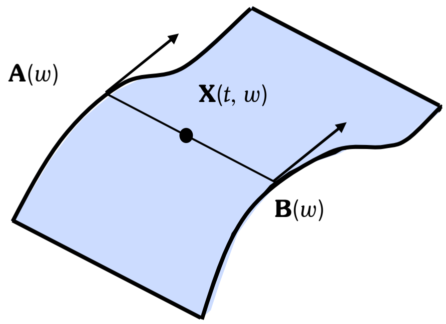

Fig. 1. A ruled surface patch

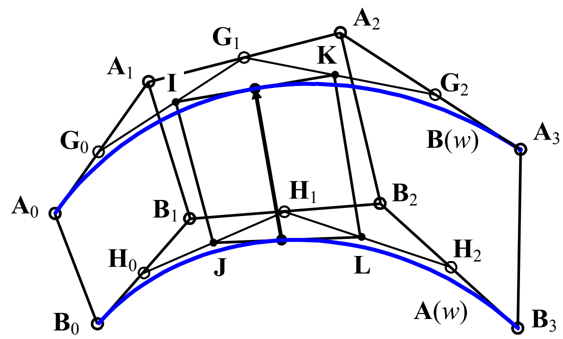

Fig. 2. The de Casteljau subdivision for a cubic Bézier patch

degrees of freedom (DOF) for the surface design. Correct design 'handles' cannot be identified that match available DOF's and fully utilize them for the surface design. Our recent findings [11] provide effective solutions for all these problems. It characterizes the degrees of freedom for the patch design, provides useful design methods without limitations in surface modeling, and derives simpler solutions for quick implementation. This paper extends these findings into geometric design of developable composite Bézier surfaces. Adjacent developable patches are joined along their end rulings while maintaining various geometric and parametric continuities across the patch boundaries. The DOF's are generalized for developable surfaces constructed with m consecutive patches of degree n subject to continuity conditions including G^0, G^1, G^2, C^1, and C^2. Design methods are proposed that make most use of the DOF's in illustrative examples. The conditions for the developability of degenerate triangular Bézier patch are also investigated. This work provides a theoretical foundation for systematic implementation of CAGD systems for developable Bézier surfaces. The results are readily extensible to the design of developable B-spline surfaces.

2. DEVELOPABILITY CONSTRAINTS IN BEZIER PATCHES

Given any two Bézier boundary curves $\mathbf{A}(w)$ and $\mathbf{B}(w)$, a ruled Bézier surface is constructed by connecting each pair of corresponding points (with equal w) with a straight line segment **AB** as shown in Fig. 1. The line segment AB is referred to as the ruling at parameter value w. The surface is expressed as: $\mathbf{X}(t, w) = (1-t)\,\mathbf{A}(w) + t\,\mathbf{B}(w)$, $0 \le t \le 1$ and $0 \le w \le 1$, where t is the parameter along the rulings. If these tangent lines and the corresponding ruling remain coplanar at every w, then the surface becomes developable. Co-planarity can be represented in terms of the triple scalar product of the two tangent vectors and the ruling vector $\mathbf{A}(w)-\mathbf{B}(w)$:

$$\dot{\mathbf{A}}(w) \times \dot{\mathbf{B}}(w) \bullet [\mathbf{A}(w)-\mathbf{B}(w)] = 0 \tag{1}$$

Substituting the Bézier representation of both curves into the above equation leads to a complicated system of equations that must be fulfilled by the Bézier control points to ensure its developability. All previous studies [4-10] impose Eqn. (1). directly on the control net of a surface, and thus non-linear systems of equations occur in the solution process. We have developed a new approach [11] to computing the constrained control points instead of solving them directly from Eqn. (1).. The developability condition is derived geometrically from the de Casteljau construction process of a Bézier patch. Cubic Bézier curves $\mathbf{A}(w)$ and $\mathbf{B}(w)$ contain control points $\mathbf{A}_0$-$\mathbf{A}_1$-$\mathbf{A}_2$-$\mathbf{A}_3$ and $\mathbf{B}_0$-$\mathbf{B}_1$-$\mathbf{B}_2$-$\mathbf{B}_3$, respectively, as shown in Fig. 2. Any value of w defines a quadratic Bézier control structure $\mathbf{G}_0$-$\mathbf{G}_1$-$\mathbf{G}_2$ and $\mathbf{H}_0$-$\mathbf{H}_1$-$\mathbf{H}_2$ with

$$\mathbf{G}_i = (1-w)\mathbf{A}_i + w\mathbf{A}_{i+1} \text{ for i = 0, 1, and 2}$$
$$\mathbf{H}_i = (1-w)\mathbf{B}_i + w\mathbf{B}_{i+1} \text{ for i = 0, 1, and 2} \tag{2}$$

For this Bézier patch, line segments **IK** and **JL** lie in the tangent direction at $\mathbf{A}(w)$ and $\mathbf{B}(w)$. Thus the developability condition indicates that **I**, **J**, **K**, and **L** lie in the same plane, and Eqn. (1). can be written as:

$$\mathbf{IJ} \bullet \mathbf{KL} \times \mathbf{IK} = 0 \tag{3}$$

I, **J**, **K** and **L** are now written as $\mathbf{I} = (1-w)\mathbf{G}_0 + w\mathbf{G}_1$, $\mathbf{J} = (1-w)\mathbf{H}_0 + w\mathbf{H}_1$, $\mathbf{K} = (1-w)\mathbf{G}_1 + w\mathbf{G}_2$, $\mathbf{L} = (1-w)\mathbf{H}_1 + w\mathbf{H}_2$, and,

$$\begin{aligned}\mathbf{IJ} &= (1-w)(\mathbf{H}_0-\mathbf{G}_0) + w(\mathbf{H}_1-\mathbf{G}_1)\\ &= (1-w)^2\mathbf{c}_0 + 2w(1-w)\mathbf{c}_1 + w^2\mathbf{c}_2\end{aligned}$$

$\mathbf{KL} = (1-w)(\mathbf{H}_1-\mathbf{G}_1) + w(\mathbf{H}_2-\mathbf{G}_2)$
$= (1-w)^2\mathbf{c}_1 + 2w(1-w)\mathbf{c}_2 + w^2\mathbf{c}_3$

$\mathbf{IK} = (1-w)(\mathbf{G}_1-\mathbf{G}_0) + w(\mathbf{G}_2-\mathbf{G}_1)$
$= (1-w)^2\mathbf{a}_1 + 2w(1-w)\mathbf{a}_2 + w^2\mathbf{a}_3$
(4)
where $\mathbf{a}_i = \mathbf{A}_i - \mathbf{A}_{i-1}$ for i = 1, 2, 3 and $\mathbf{c}_j = \mathbf{B}_j - \mathbf{A}_j$ for j = 0, 1, 2, 3.

Substituting Eqn. (4) into Eqn (3) results in a univariate polynomial in w of degree six. The coefficients of the polynomial must vanish for any w. Thus seven constraints are obtained that must impose on the Bézier control points:

$$\mathbf{a}_1\bullet\mathbf{c}_0\times\mathbf{c}_1 = 0 \quad (5)$$

$$\mathbf{a}_3\bullet\mathbf{c}_2\times\mathbf{c}_3 = 0 \quad (6)$$

$$\mathbf{a}_1\bullet\mathbf{c}_0\times\mathbf{c}_2 + \mathbf{a}_2\bullet\mathbf{c}_0\times\mathbf{c}_1 = 0 \quad (7)$$

$$\mathbf{a}_2\bullet\mathbf{c}_2\times\mathbf{c}_3 + \mathbf{a}_3\bullet\mathbf{c}_1\times\mathbf{c}_3 = 0 \quad (8)$$

$$\mathbf{a}_1\bullet\mathbf{c}_0\times\mathbf{c}_3 + 3\mathbf{a}_1\bullet\mathbf{c}_1\times\mathbf{c}_2 + 4\mathbf{a}_2\bullet\mathbf{c}_0\times\mathbf{c}_2 + \mathbf{a}_3\bullet\mathbf{c}_0\times\mathbf{c}_1 = 0 \quad (9)$$

$$\mathbf{a}_1\bullet\mathbf{c}_2\times\mathbf{c}_3 + 4\mathbf{a}_2\bullet\mathbf{c}_1\times\mathbf{c}_3 + \mathbf{a}_3\bullet\mathbf{c}_0\times\mathbf{c}_3 + 3\mathbf{a}_3\bullet\mathbf{c}_1\times\mathbf{c}_2 = 0 \quad (10)$$

$$\mathbf{a}_1\bullet\mathbf{c}_1\times\mathbf{c}_3 + \mathbf{a}_2\bullet\mathbf{c}_0\times\mathbf{c}_3 + 3\mathbf{a}_2\bullet\mathbf{c}_1\times\mathbf{c}_2 + \mathbf{a}_3\bullet\mathbf{c}_0\times\mathbf{c}_2 = 0 \quad (11)$$

Note that Eqn. (5) indicates that the first two pairs of control points $\mathbf{A}_0$-$\mathbf{B}_0$ and $\mathbf{A}_1$-$\mathbf{B}_1$ lie in the same plane. The last two pairs of control points $\mathbf{A}_2$-$\mathbf{B}_2$ and $\mathbf{A}_3$-$\mathbf{B}_3$ are also coplanar because of Eqn. (6). These two constraints are referred to as the co-planarity condition.

The developability condition Eqn. (3) must hold for a Bézier patch of any degree. Suppose the degree of the boundary curves is n. The coordinates of **I**, **J**, **K** and **L** are (n–1)-degree polynomials in the curve parameter w. Vectors **IJ**, **KL**, and **IK** also have coefficients that are (n–1)-degree polynomials in w. Substituting them into Equation (3) results in a 3(n–1)-degree polynomial with (3n–2) coefficients that must vanish for any w. Alternatively stated, there are (3n–2) independent constraints to fulfill the developability condition. The first as well as the second boundary curves, each has (n+1) control points in 3D space, contributing 3(n+1) degrees of freedom. After the first curve has been specified, the degrees of freedom available for the second one are 3(n+1) – (3n–2) = 5, independent of the degree of the patch.

3. CONSTRAINTS IN DEVELOPABLE COMPOSITE BEZIER SURFACES

A developable composite Bézier surface is constructed by joining consecutive patches along their end rulings. In addition to the developability constraints of each patch, the control points of the surface must satisfy certain constraints to maintain continuities across the patch boundary. The degrees of freedom available for the surface design are thus reduced. Suppose the user has specified the first boundary curve with a given continuity condition. It is advantageous to determine how many free design parameters remain in the control net of the second curve. We first examine a cubic developable surface consisting of two adjacent patches with various geometric and parametric continuities. The results are then generalized for a surface consisting of m patches of degree n.

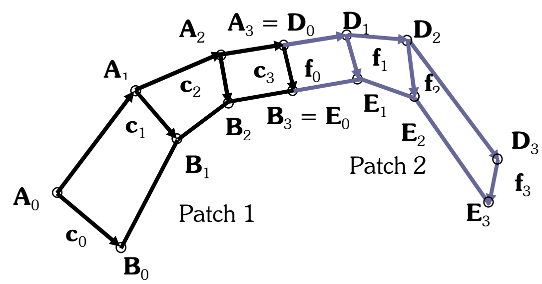

Fig. 3. Control polygon of a developable composite Bézier surface of degree three

3.1 Geometric Continuity

After a first boundary curve has been chosen, there are eight control points in 3D space to be determined for the other curve. Each patch must satisfy its seven developability constraints, Eqns. (5-11). Therefore, the number of the remaining DOF's is 8(3) – 7 – 7 = 10. This simply reflects the conclusion that each developable Bézier patch has five DOF's in general. Fig. 3 shows that the last control point of the first patch must coincide with the first control point of the second patch due to positional continuity, i.e. $\mathbf{B}_3 = \mathbf{E}_0$, using up three DOF's. As a result, seven DOF's are available for the design of the second curve. For gradient continuity, the tangent vector of the second patch must be collinear with that of the first patch at the end point, which is written as:

$$\mathbf{B}_2\mathbf{B}_3 = \mu\mathbf{E}_0\mathbf{E}_1 \quad (12)$$

where μ is the length ratio of the two tangent vectors. Note that $\mathbf{A}_2$, $\mathbf{A}_3$=$\mathbf{D}_0$, $\mathbf{D}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$=$\mathbf{E}_0$, and $\mathbf{E}_1$, lie in the same plane due to the co-planarity condition. Hence to impose Eqn. (12) only consumes one DOF and thus six DOF's remain for the surface design. The curvature continuity requires that $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$=$\mathbf{E}_0$, $\mathbf{E}_1$, and $\mathbf{E}_2$ are coplanar [12]. Since $\mathbf{E}_0$ and $\mathbf{E}_1$ have already lied in the plane because of the gradient continuity, this imposes one more constraint on $\mathbf{E}_2$: it must be located in the plane determined by $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$. Consequently five DOF's are left in this case.

We generalize the above results for a developable composite surface consisting of m adjacent patches of degree n. Section 2 has shown that a developable Bézier patch of degree n has (3n–2) constraints in specifying the second boundary curve after the first one has been chosen. Thus the composite surface containing m patches must satisfy totally m(3n–2) equations to ensure its developability. Any two consecutive patches impose three more constraints on their common boundary for the positional continuity. There are (m–1) such boundaries in the surface, summing up to 3(m–1) constraints. The second boundary curve has m(n+1) control points, contributing 3m(n+1) degrees of freedom. The number of DOF's thus becomes 3m(n+1) – m(3n–2) – 3(m–1) = 2m+3.

To ensure the gradient continuity across each boundary gives additional (m–1) constraints. The degrees of freedom for the surface design are further reduced to (2m+3) – (m–1) = m+4. The curvature continuity across the patch boundaries imposes another (m–1) constraints. Therefore, the degrees of freedom for a developable composite Bézier surface with G^2 continuity become (m+4) – (m–1) = 5, regardless of the number of the comprising patches and the degree of the surface. Tab. 1 summarizes the results for G^0, G^1, and G^2 continuities.

Continuity	Number of Degrees of Freedom
G^0	*2m+3*
G^1	*m+4*
G^2	*5*

Tab. 1. Available degrees of freedom for various geometric continuities (m: number of patches)

3.2 Parametric Continuity

As derived previously, seven DOF's are available for the design of the second boundary curve that satisfies positional continuity. The first derivative continuity requires:

$$\mathbf{E}_1 - \mathbf{E}_0 = \mathbf{B}_3 - \mathbf{B}_2 \quad (13)$$

which uses up only two degrees of freedom. Notice that to specify the relationship of two vectors according to Eqn. (13) adds two more constraints, not three. Hence only five DOF's remain in this case. The second derivative continuity fixes the position of the third control point $\mathbf{E}_2$ [13]:

$$\mathbf{E}_2 - \mathbf{B}_1 = 4(\mathbf{B}_3 - \mathbf{B}_2) \quad (14)$$

Similarly, this equation specifies the relative position of two vectors, consuming two more degrees of freedom for the control polygon. Only three DOF's are left in this case. A developable Bézier patch of degree n has (3n–2) constraints in specifying the second boundary curve after the first one has been chosen. As stated previously, only (2m+3) DOF's are available in order to maintain the positional continuity across m patches. The first derivative continuity imposes two constraints on each common boundary, for totally (m–1) boundaries. The remaining DOF's are computed as (2m+3) – 2(m–1) = 5, independent of the degree of the surface. The second derivative requires two more constraints to be satisfied across each boundary. The DOF's left for the surface design is 5 – 2(m–1) = 7 – 2m. The result indicates that such a surface consists of at most three patches, regardless of the degree of the surface. Tab. 2 summarizes the corresponding DOF's for C^0, C^1, and C^2 continuities

Continuity	Number of Degrees of Freedom
C^0	*2m+3*
C^1	*5*
C^2	*7-2m*

Tab. 2. Available degrees of freedom for various parametric continuities (m: number of patches)

4. DEGENERATE DEVELOPABLE COMPOSITE BEZIER SURFACES

Topologically triangular patches are often needed to create a desired shape. A three-sided patch is constructed by allowing the end points of two boundary curves to coincide. The number of DOF's is changed for the design of such a degenerate surface, but it must still satisfy the Eqn. (3), $\mathbf{IJ} \bullet \mathbf{KL} \times \mathbf{IK} = 0$. With the boundary curves of degree n, each $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{L}$ can be expressed as a polynomial in curve parameter w of degree (n–1):

$$\mathbf{IJ} = \sum_{i=0}^{n-1} C_i^{n-1} w^i (1-w)^{(n-1)-i} \mathbf{c}_i$$

$$\mathbf{KL} = \sum_{i=0}^{n-1} C_i^{n-1} w^i (1-w)^{(n-1)-i} \mathbf{c}_{i+1} \quad (15)$$

$$\mathbf{IK} = \sum_{i=0}^{n-1} C_i^{n-1} w^i (1-w)^{(n-1)-i} \mathbf{a}_{i+1}$$

where $C_i^{n-1} = \dfrac{(n-1)!}{(n-1-i)!\cdot i!}$ and $\mathbf{a}_i = \mathbf{A}_i - \mathbf{A}_{i-1}$, $\mathbf{c}_i = \mathbf{B}_i - \mathbf{A}_i$ for i = 0, 1, … n. $\mathbf{A}_i$ and $\mathbf{B}_i$ are the control points of the boundary curves. Since the end points coincide in a triangular patch, $\mathbf{KL}$ becomes:

$$\mathbf{KL} = \sum_{i=0}^{n-2} C_i^{n-1} w^i (1-w)^{(n-1)-i}\, \mathbf{c}_{i+1} \tag{16}$$

Substituting the above expressions into Eqn. (3) leads to:

$$[\sum_{i=0}^{n-1} C_i^{n-1} w^i (1-w)^{(n-1)-i}\, \mathbf{a}_{i+1}]\cdot$$

$$[\sum_{i=0}^{n-1} C_i^{n-1} w^i (1-w)^{(n-1)-i}\, \mathbf{c}_{i}] \times$$

$$[\sum_{i=0}^{n-2} C_i^{n-1} w^i (1-w)^{(n-1)-i}\, \mathbf{c}_{i+1}]$$

$$=[C_0^{n-1}(1-w)^{n-1}\,\mathbf{a}_1 + C_1^{n-1} w(1-w)^{n-2}\,\mathbf{a}_2 + \ldots$$

$$+ C_{n-1}^{n-1} w^{n-1}\,\mathbf{a}_n]\cdot$$

$$[C_0^{n-1}(1-w)^{n-1}\,\mathbf{c}_0 + C_1^{n-1} w(1-w)^{n-2}\,\mathbf{c}_1 + \ldots$$

$$+ C_{n-2}^{n-1} w^{n-2}(1-w)\,\mathbf{c}_{n-2} + C_{n-1}^{n-1} w^{n-1}\,\mathbf{c}_{n-1}] \times$$

$$[C_0^{n-1}(1-w)^{n-1}\,\mathbf{c}_1 + C_1^{n-1} w(1-w)^{n-2}\,\mathbf{c}_2 + \ldots$$

$$+ C_{n-3}^{n-1} w^{n-3}(1-w)^2\,\mathbf{c}_{n-2} + C_{n-2}^{n-1} w^{n-2}(1-w)\,\mathbf{c}_{n-1}]$$

$$=[C_0^{n-1}(1-w)^{n-1}\,\mathbf{a}_1 + C_1^{n-1} w(1-w)^{n-2}\,\mathbf{a}_2 + \ldots$$

$$+ C_{n-1}^{n-1} w^{n-1}\,\mathbf{a}_n]\cdot$$

$$[C_0^{n-1} C_0^{n-1}(1-w)^{2n-2}\,\mathbf{c}_0 \times \mathbf{c}_1$$

$$+ C_0^{n-1} C_1^{n-1} w(1-w)^{2n-3}\,\mathbf{c}_0 \times \mathbf{c}_2 + \ldots$$

$$+ C_{n-1}^{n-1} C_{n-3}^{n-1} w^{2n-4}(1-w)^2\,\mathbf{c}_{n-1} \times \mathbf{c}_{n-2}$$

$$+ C_{n-2}^{n-1} C_{n-2}^{n-1} w^{2n-4}(1-w)^2\,\mathbf{c}_{n-1} \times \mathbf{c}_{n-2}$$

$$+ C_{n-1}^{n-1} C_{n-2}^{n-1} w^{2n-3}(1-w)\,\mathbf{c}_{n-1} \times \mathbf{c}_{n-1}]$$

where the last term vanishes due to $\mathbf{c}_{n-1} \times \mathbf{c}_{n-1} = 0$. As a result, each term in the second bracket contains $(1-w)^2$. When $w \neq 1$, dividing the above equation by $(1-w)^2$ leads to:

$$[C_0^{n-1}(1-w)^{n-1}\,\mathbf{a}_1 + C_1^{n-1} w(1-w)^{n-2}\,\mathbf{a}_2 + \ldots$$

$$+ C_{n-1}^{n-1} w^{n-1}\,\mathbf{a}_n]$$

$$[C_0^{n-1} C_0^{n-1}(1-w)^{2n-4}\,\mathbf{c}_0 \times \mathbf{c}_1$$

$$+ C_0^{n-1} C_1^{n-1} w(1-w)^{2n-5}\,\mathbf{c}_0 \times \mathbf{c}_2 + \ \ldots$$

$$+ C_{n-1}^{n-1} C_{n-3}^{n-1} w^{2n-4}\,\mathbf{c}_{n-1} \times \mathbf{c}_{n-2}$$

$$+ C_{n-2}^{n-1} C_{n-2}^{n-1} w^{2n-4}\,\mathbf{c}_{n-1} \times \mathbf{c}_{n-2}] = 0 \tag{17}$$

which becomes a polynomial in w of degree (3n–5). In order to satisfy the developability condition, all coefficients must equal zero, imposing (3n–4) constraints on the control polygon of the patch. If the first boundary curve is freely chosen, then there are totally n control points to be specified for the design of the second curve, as the end points of the patches superimpose on one side. The available degrees of freedom become 3n – (3n–4) = 4. Hence, merely four additional DOF's are available for the design of a degenerate developable Bézier patch with one end ruling vanishing. Similar derivations can be conducted on a patch with both end rulings vanishing. The corresponding developability condition becomes a polynomial in w of degree (3n–7) with (3n–6) coefficients that must equal zero. The second boundary curve has only (n–1) control points to be determined, since both end points of the first curve must coincide with those of the second curve. There remain 3(n–1) – (3n–6) = 3 degrees of freedom for the design of the second curve, regardless of the degree of the patch.

5. DESIGN EXAMPLES

A Bézier surface comprised of two cubic developable patches is used as a test example to verify the derived results. There are various ways to use available degrees of freedom in the surface design, each of which has different computational requirement in solving the constrained control points. This paper will not address the advantages of one particular design method over another. Instead, the focus is to demonstrate how to identify feasible "design handles" with the limited DOF's that do not induce an over-constrained system in the solution process. The first and second boundary curves are referred to as the A-curve and B-curve respectively in the following examples.

G^0 continuity: the A-curve is constructed with $\mathbf{A}_0$(-15, 0, 5), $\mathbf{A}_1$(-5, 10, 5), $\mathbf{A}_2$(5, 12, 3), $\mathbf{A}_3$(15, 2, 7), $\mathbf{D}_0$(15, 2, 7), $\mathbf{D}_1$(20, -3, 10), $\mathbf{D}_2$(23, -8, 11), and **D**3(27, -13, 13). The user is allowed to specify $\mathbf{B}_0$(-6, 0, 2) and $\mathbf{B}_1$(-2, 4, 2) of the first patch, consuming three and two DOF's respectively. The remaining two DOF's are used to place the control point $\mathbf{E}_1$ of the second patch at (12, -10.3, 7). Fig. 4 illustrates one resultant developable surface solved from the system with $\mathbf{B}_2$(2, 4.8, 1.2), $\mathbf{B}_3$(6, 0.8, 2.8), $\mathbf{E}_2$(14.3, -24.45, 7.6), and $\mathbf{E}_3$(18.35, -43.175, 10.1). The first and second patches are shown in gray and lavender respectively. Note that multiple solution may exist that all satisfy the developability constraints of the first patch [11].

G^1 continuity: the A-curve is chosen as $\mathbf{A}_0$(-15, 0, 5), $\mathbf{A}_1$(-5, 10, 5), $\mathbf{A}_2$(5, 12, 3), $\mathbf{A}_3$(15, 2, 7), $\mathbf{D}_0$(15, 2, 7), $\mathbf{D}_1$(20, -3, 9), $\mathbf{D}_2$(23, 0, 8), and $\mathbf{D}_3$(27, -5, 10). To place $\mathbf{B}_0$(6, 0.8, 2.8) and $\mathbf{B}_1$(14, -7.2, 6) for the first patch consumes three and two DOF's respectively. $\mathbf{B}_2$(2, 4.8, 1.2) and $\mathbf{B}_3$(6, 0.8, 2.8) are fully determined after these five DOF's have been used up. The last DOF is used to locate the position of $\mathbf{E}_1$ at (14, -7.2, 6) along the $\mathbf{B}_2\mathbf{B}_3$ direction. Fig. 5 shows one resulting surface with $\mathbf{E}_2$(18.8, -2.4, 4.4) and $\mathbf{E}_1$(25.2, -10.4, 7.6).

G^2 continuity: the A-curve is chosen as $\mathbf{A}_0$(-15, 0, 5), $\mathbf{A}_1$(-5, 10, 5), $\mathbf{A}_2$(5, 12, 3), $\mathbf{A}_3$(15, 2, 7), $\mathbf{D}_0$(15, 2, 7), $\mathbf{D}_1$(20, -3, 9), $\mathbf{D}_2$(30, -7, 10), and $\mathbf{D}_3$(34, -12, 12). Five DOF's are used in specifying $\mathbf{B}_0$(-6, 0, 2) and $\mathbf{B}_1$(-2, 4, 2) for the first patch. The other control points are automatically determined by the system. One solution set is obtained with $\mathbf{B}_2$(2, 4.8, 1.2), $\mathbf{B}_3$(6, 0.8, 2.8), $\mathbf{E}_0$(6, 0.8, 2.8), $\mathbf{E}_1$(14, -7.2, 6), $\mathbf{E}_2$(30, -13.6, 7.6), and $\mathbf{E}_3$(36.4, -21.6, 10.8), as shown in Fig. 6.

C^1 continuity: the A-curve is specified as $\mathbf{A}_0$(-15, 0, 5), $\mathbf{A}_1$(-5, 10, 5), $\mathbf{A}_2$(5, 12, 3), $\mathbf{A}_3$(15, 2, 7), $\mathbf{D}_0$(15, 2, 7), $\mathbf{D}_1$(25, -8, 11), $\mathbf{D}_2$(28, -5, 10), and $\mathbf{D}_3$(32, -10, 12). Five DOF's are consumed in specifying $\mathbf{B}_0$(-6, 0, 2) and $\mathbf{B}_1$(-2, 4, 2) for the first patch. $\mathbf{B}_2$(2, 4.8, 1.2) and $\mathbf{B}_3$(6, 0.8, 2.8) are then automatically determined by the system. $\mathbf{E}_0$(6, 0.8, 2.8) and $\mathbf{E}_1$(10, -3.2, 4.4) are also fully defined because of C^1 continuity. The remaining control points are computed as $\mathbf{E}_2$(11.2, -2, 4) and $\mathbf{E}_3$(12.8, -4, 4.8). The resulting surface is shown in Fig. 7.

C^2 continuity: only three DOF's are available in specifying the B-curve. The user may want to choose $\mathbf{B}_0$, consuming all these DOF's. In this case, the remaining six control points (totally eighteen coordinate values in 3D space) must be solved simultaneously from the system. To simplify the solution process, we let the system determine the last control point $\mathbf{D}_3$ and thus gain three extra DOF's. The user cannot freely place this point any more. The A-curve contains $\mathbf{A}_0$(-15, 0, 5), $\mathbf{A}_1$(-5, 10, 5), $\mathbf{A}_2$(5, 12, 3), $\mathbf{A}_3$(15, 2, 7), $\mathbf{D}_0$(15, 2, 7), $\mathbf{D}_1$(25, -8, 11), and $\mathbf{D}_2$(35,-30,21). The three gained DOF's are used to determine $\mathbf{B}_0$(-6, 0, 2). To specify $\mathbf{B}_1$(-2, 4, 2) consumes the remaining two DOF's. $\mathbf{B}_2$(2, 4.8, 1.2) and $\mathbf{B}_3$(6, 0.8, 2.8) are determined by the developability constraints. $\mathbf{E}_0$(6, 0.8, 2.8), $\mathbf{E}_1$(10, -3.2, 4.4), and $\mathbf{E}_2$(14, -12, 8.4) of the second patch are also fully defined because of the C^0, C^1, and C^2 conditions. The user can choose the length of $\mathbf{D}_3\mathbf{E}_2$ as 10 with the last DOF. The system will determine $\mathbf{D}_3$(43.373, -33.333, 16.667) and $\mathbf{E}_3$(17.349, -13.333, 6.667). The final surface is shown in Fig. 8.

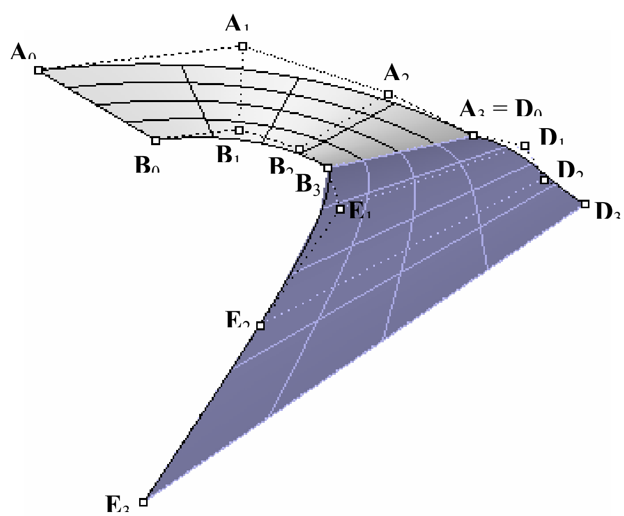

Fig. 4. Design example of G^0 continuity

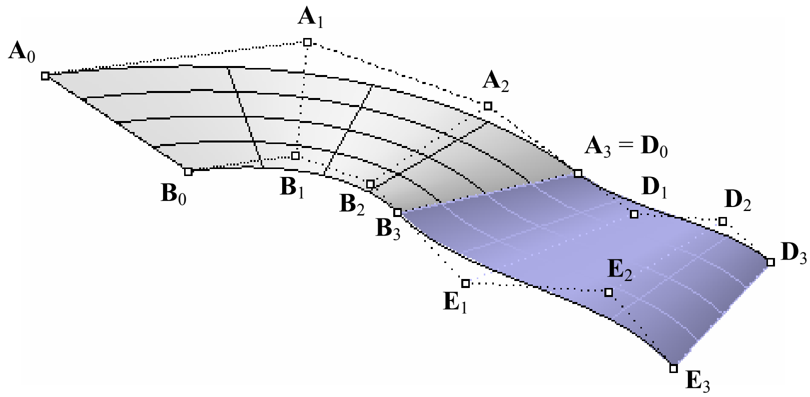

Fig. 5. Design example of G^1 continuity

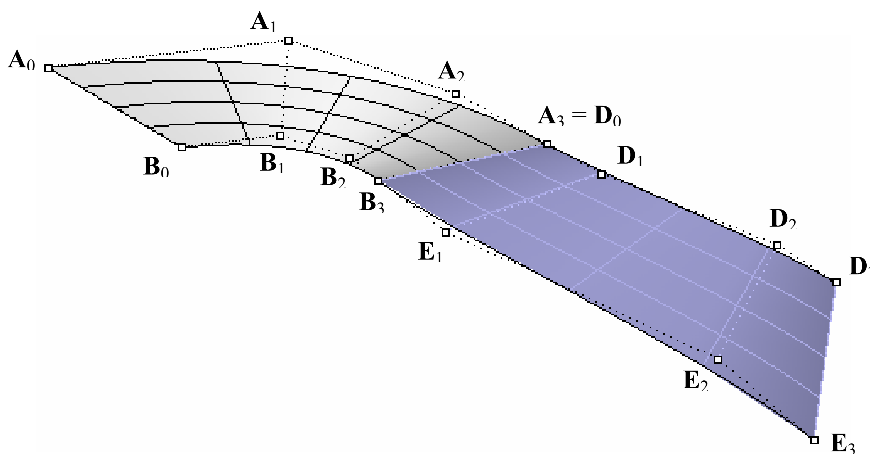

Fig. 6. Design example of G^2 continuity

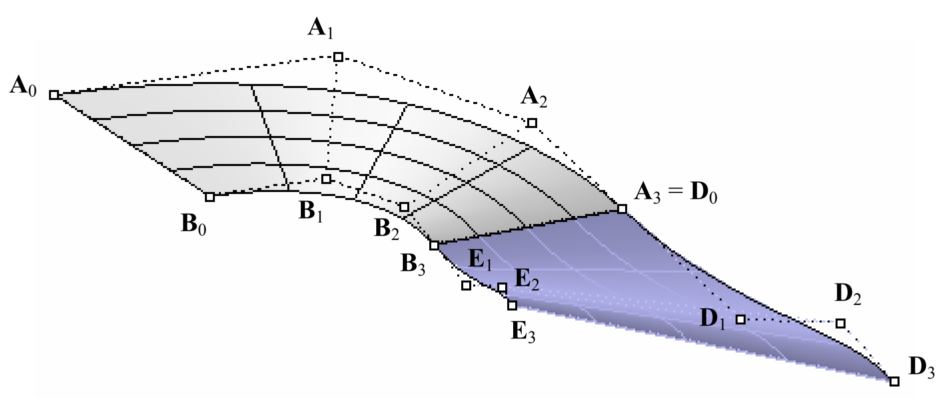

Fig. 7. Design example of C^1 continuity

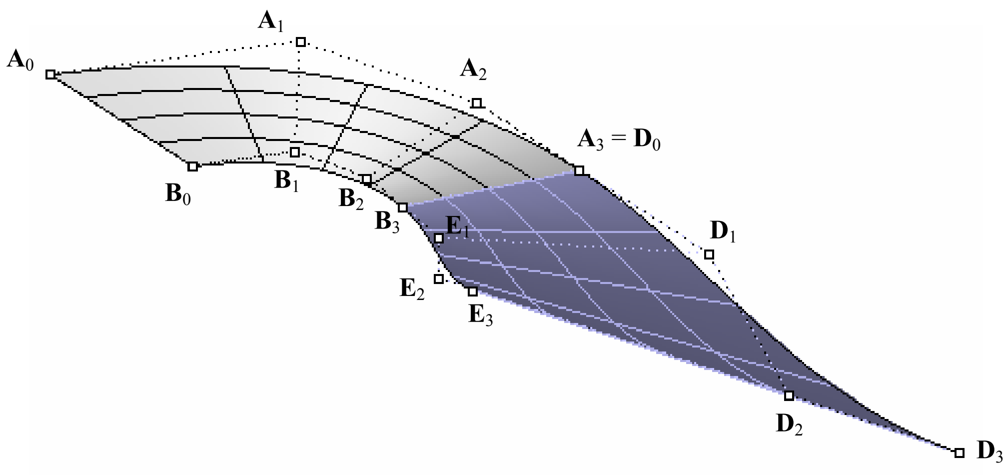

Fig. 8. Design example of C^2 continuity

One-end degenerate patch: the A-curve contains $\mathbf{A}_0$(-10, -10, 0), $\mathbf{A}_1$(0, 0, 0), $\mathbf{A}_2$(10, 2, -2), and $\mathbf{A}_3$(20, 2, -12). To specify $\mathbf{B}_0$(0, -10, 0) consumes three DOF's. The last DOF can be used to place $\mathbf{B}_1$ along a given direction (1, -1, 0). The length of the vector $\mathbf{A}_1\mathbf{B}_1$ and other control points are determined by the system: $\mathbf{B}_1$(11.67, -11.67, 0), $\mathbf{B}_2$(15.833, 10.167, 0.333), and $\mathbf{B}_3 = \mathbf{A}_3$. Fig. 9 shows the resulting patch.

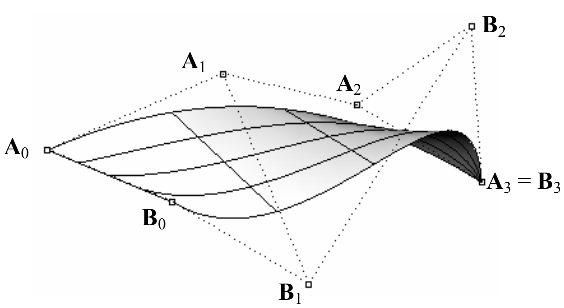

Fig. 9. One-end degenerate cubic patch

Both-end degenerate patch: the A-curve contains $\mathbf{A}_0$(-5, -10, 0), $\mathbf{A}_1$(0, 0, 0), $\mathbf{A}_2$(10, 2, -2), and $\mathbf{A}_3$(20, 2, -12). To specify $\mathbf{B}_1$(0, -10, 0) consumes only two DOF's, because $\mathbf{A}_1$-$\mathbf{A}_2$-$\mathbf{B}_1$-$\mathbf{B}_2$ lie in the same plane. The last DOF is used to define the length of $\mathbf{A}_2\mathbf{B}_2$ (3 in this case). $\mathbf{B}_2$(10, 5, -2) is then solved from the system. The resulting patch is shown in Fig. 10.

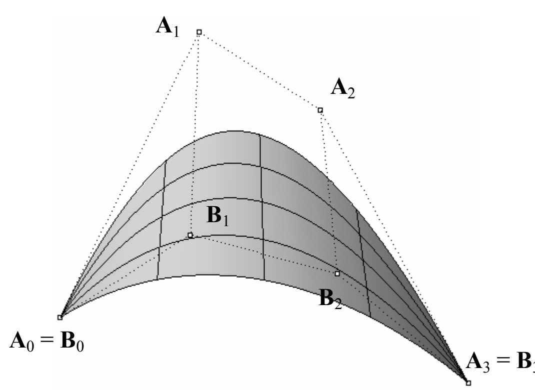

Fig. 10. Both-end degenerate cubic patch

6. DISCUSSION AND CONCLUSIONS

This study investigates geometric design of developable surfaces that consist of consecutive Bézier patches joined along the end rulings. The developability condition has been combined with the de Casteljau algorithm, resulting in a set of constraints that must be fulfilled by the control polyhedron of each patch. Additional constraints are imposed on the patch boundaries due to various continuity conditions. Suppose a developable surface of degree n is comprised of m developable Bézier patches. The user can freely choose a first boundary curve. For a secondary boundary curve of the same degree, there remain (2m+3), (m+4), and five degrees of freedom for the design of the surface that is G^0, G^1, and G^2 respectively. The degrees of freedom become five and (7–2m) when the surface is C^1 and C^2. The results are independent of the degree of the surface. Only four degrees of freedom are left for the design of a degenerate developable Bézier patch constructed by allowing one pair of the end points to superimpose. Three degrees of freedom remain if the end rulings on both sides of the control net vanish. These conclusions are also regardless of the degree of the surface. Several examples have been presented to illustrate the design of developable Bézier surfaces in which the degrees of freedom are fully utilized. The focus is to demonstrate practical design methods that provide the user intuitive design "handles" to control the surface shape while over-constrained systems are not induced in the solution process. In general there exist multiple solutions for the constrained control net. This work extends previous research on geometric design of developable Bézier patches to developable composite Bézier surfaces. It provides the foundation for an easy and, more importantly, systematic implementation of CAGD systems for developable surfaces. The results should also be readily extendable to design of developable shapes using B-spline surfaces. Our current research is focused on this.

7. ACKNOWLEDGEMENTS

This work was supported by the National Science Council of Taiwan under grant number NSC 91-2218-E-007-045. The authors would also like to express their gratitude to Professor Carlo H. Séquin at the University of California, Berkeley, for his early guidance on this work.

8. REFERENCES

[1] Ferris L.W. A standard series of developable surfaces. Marine Technology; Vol. 5, No. 1, 1968, pp 53-62.

[2] Mortenson M. Geometric modeling. John Wiley & Sons, 1985.

[3] Clements J.C. A computer system to derive developable hull surfaces and tables of offsets. Marine Technology; Vol. 18, No. 3, 1981, pp 227-233.

[4] Aumann G. Interpolation with developable Bézier patches. Computer Aided Geometric Design, Vol, 8, 1991, pp 409-420.

[5] Lang J., Röschel O. Developable (1,n)-Bézier surfaces. Computer Aided Geometric Design, Vol. 9, 1992, pp 291-298.

[6] Frey W.H., Bindschadler D. Computer aided design of a class of developable Bézier surfaces. GM Research Publication 1993, GMR-8057.

[7] Maekawa T., Chalfant J.S. Design and tessellation of B-Spline developable surfaces. ASME Transaction Journal of Mechanical Design, Vol. 120, 1998, pp 453-461.

[8] Chalfant J.S., Maekawa T. Design for manufacturing using B-Spline developable surfaces. Journal of Ship Production, Vol. 42, No. 3, 1998, pp 207-215.

[9] Bodduluri R.M.C., Ravani B. Geometric design and fabrication of developable Bézier and B-spline surfaces. ASME Transaction Journal of Mechanical Design, Vol. 116, 1994, pp 1024-1048.

[10] Ravani B., Wang J.W. Computer aided geometric design of line constructs. ASME Journal of Mechanical Design, Vol. 113, 1991, pp 363-271.

[11] Chu C.H., Séquin C.H. Developable Bézier Patches: Properties and Design. Computer Aided Design, Vol. 34, No. 7, 2002, pp 511-527.

[12] Faux I.D., Pratt M.J. Computational Geometry for Design and Manufacture, Ellis Horwood, 1978.

[13] Bartels R.H., Beatty J.C., Barsky B.A. An introduction to splines for use in computer graphics and geometric modeling, Morgan Kaufmann, 1987.