Algebraic Algorithms for Computing Intersections between Torus and Natural Quadrics

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ABSTRACT

We present in this paper efficient and robust algebraic algorithms for computing intersection curves between torus and natural quadrics used in CAD, namely sphere, cylinder and cone. A proper local coordinate system (LCS) is chosen first. Under the chosen LCS, cylinders and cones are parametrized by their line generators, and the problem of computing intersections between tori and cylinders or cones are converted to the line/torus intersection problem. Then, cases that the intersections are conic sections are identified and computed in geometrical and intuitive ways. For all other cases, discrete intersection points are computed and sorted to form intersection curves. For torus/sphere intersection, we present algorithms to compute the intersection curves directly from explicit algebraic representations.

Keywords: CAD; Cylinder; Cone; Sphere; Torus; Intersection; Non-planar Section; Sorting

1. INTRODUCTION

Quadric surfaces, such as cylinders and cones, and tori are basic geometric elements in CAD systems since they are the most widely used geometric primitives in mechanical parts. In the boundary evaluation of such mechanical parts, we need to compute the intersection curves of these surfaces. How to efficiently and robustly compute intersection curves between these surfaces is critical to the robustness and performances of the CAD systems. There had been already considerable amount of results in this field [1-10]. For example, Levin[3] suggested to classify and parameterize quadrics and to compute the intersections by algebraic approach; Sarraga[2] presented an effective approach to compute quadric surface intersections.

While researches on efficient and robust computation of quadric surface intersections algebraically have been fairly mature [1-6], there are few researches on algebraic approaches for torus and quadric intersections. Currently, intersections between torus and quadrics in CAD systems are computed mostly by intersections between NURBS representations of the torus and quadrics. This way, the numeric precision is not easy to control, and the computational stability is not easy to ensure. Piegl[7] considered a case that a torus has two tangential intersections with a plane. In this case, the intersection curves consist of two circles (called Yvone-Villiaroeau circles). Kim et. al. [8-10] presented the conditions under which the intersections between torus and quadrics are conic sections. However, direct algorithms for computing non-planar section intersections between torus and quadrics are missing.

We present in this paper efficient and robust algebraic algorithms for computing intersection curves between torus and natural quadrics used in CAD, namely sphere, cylinder and cone. Our algorithms start with setting a proper local coordinate system (LCS). Under the chosen LCS, the torus has a standard form, and the quadrics have relatively simple forms. Cylinders and cones are parametrized by their line generators, and the problem of computing intersections between tori and cylinders or cones is converted to the line/torus intersection problem, and then it can be characterized by a quartic (degree 4) algebraic equation. To ensure the robustness and performance efficiency of the downstream operations, CAD systems should be as analytic as possible. That is to say, analytic intersection curves should be extracted directly from surface configurations. We present a geometric and intuitive way to identify and compute the conic section intersections. For those that the intersections are not conic sections, unordered intersection points are computed by solving the quartic equation to a prescribed precision. We present a novel, efficient and robust algorithm to sorting these unordered intersection points to form intersection curves. For torus/sphere intersection, the intersection curves can be explicitly represented by algebraic forms, and we present such forms in the paper.

The rest of the paper is organized as follows. Section 2 presents the quartic equations governing the torus/cylinder and torus/cone intersections with the cylinder or cone parametrized by its line generators. Presented in this section also the algorithm to solve the equations to obtain discrete intersection points. It then presents geometric and intuitive ways to identify and compute conic section intersections. Section 3 presents an efficient and robust algorithm to sort the obtained discrete intersection points. Section 4 presents an algorithm to compute the intersections between torus and sphere. These intersection curves have explicit algebraic expressions. Finally, in section 5, we conclude this paper.

2. TORUS AND CYLINDER OR CONE INTERSECTIONS

To obtain the intersection curves between torus and cylinder or cone, we first set a LCS properly based on the axes of the torus and the cylinder or the cone, in order to obtain simple geometric formulae. Under the LCS, the torus can have a standard form. Next, we parametrize the cylinder or cone using its line generators: a set of straight lines parametrized by variable t with respect to the circular angle θ. The intersection curves can then be characterized by a quartic equation about the variable t for a fixed angle θ. From this equation, we can accurately compute t by solving the equation with coefficients with respect to θ. We can easily control discretizing precision by adjusting the incremental value $\Delta\theta$ of angle θ. We also present explicit intersection curves geometrically and intuitively if the intersection curves happen to be conic sections.

2.1 Torus and Cylinder Intersections

Suppose that torus S_0 has a center P_0, axis d irection $\overline{A_0}$ ($|\overline{A_0}|=1$), major radius R_0, and minor radius r_0. Suppose that cylinder S_1 has an axis point P_1, axis direction $\overline{A_1}$ ($|\overline{A_1}|=1$), and radius R_1.

In the following, we will choose a proper LCS to obtain simple forms for the torus and the cylinder. We first present the case that $\overline{A_0}$ isn't parallel to $\overline{A_1}$; then the case that $\overline{A_0}$ is parallel to $\overline{A_1}$ but not collinear; and finally the case that $\overline{A_0}$ and $\overline{A_1}$ are collinear.

If $\overline{A_0}$ isn't parallel to $\overline{A_1}$, then we set the LCS{ P_0; $\overline{A_0^*}$, $\overline{A_1^*}$, $\overline{A_2^*}$ }, where, $\overline{A_0^*}=\overline{A_0}$,

$\overline{A_1^*}=\dfrac{\overline{A_0^*}\times\overline{A_1}}{|\overline{A_0^*}\times\overline{A_1}|}$, and $\overline{A_2^*}=\overline{A_0^*}\times\overline{A_1^*}$.

In this coordinate system,

$$S_0:(x^2+y^2+z^2+R_0^{\ 2}-r_0^{\ 2})^2=4R_0^{\ 2}(y^2+z^2) \qquad (2.1)$$

Let $\overline{V}=P_1-P_0=v_0\cdot\overline{A_0^*}+v_1\cdot\overline{A_1^*}+v_2\cdot\overline{A_2^*}$,

where, $v_0=\overline{V}\cdot\overline{A_0^*}$, $v_1=\overline{V}\cdot\overline{A_1^*}$, $v_2=\overline{V}\cdot\overline{A_2^*}$.

Let $\overline{B_1}=\overline{A_1^*}$ and $\overline{C_1}=\dfrac{\overline{A_1}\times\overline{A_1^*}}{|\overline{A_1}\times\overline{A_1^*}|}$, then the cylinder S_1 can be represented as follows:

$$S_1: \qquad P_1(t,\theta)=P_1+t\overline{A_1}+R_1\cos\theta\overline{B_1}+R_1\sin\theta\overline{C_1}$$

(2.2)

where,

$$\begin{cases}\overline{A_1}=a_0\overline{A_0^*}+a_1\overline{A_1^*}+a_2\overline{A_2^*}=a_0\overline{A_0^*}+a_2\overline{A_2^*}\\ \overline{B_1}=\overline{A_1^*}\\ \overline{C_1}=c_0\overline{A_0^*}+c_1\overline{A_1^*}+c_2\overline{A_2^*}=c_0\overline{A_0^*}+c_2\overline{A_2^*}\end{cases}$$

(2.3)

with

$$\begin{cases}a_0=\overline{A_1}\cdot\overline{A_0^*}\\ a_1=\overline{A_1}\cdot\overline{A_1^*}=0\\ a_2=\overline{A_1}\cdot\overline{A_2^*}\end{cases},\quad \begin{cases}c_0=\overline{C_1}\cdot\overline{A_0^*}\\ c_1=\overline{C_1}\cdot\overline{A_1^*}=0\\ c_2=\overline{C_1}\cdot\overline{A_2^*}\end{cases}$$

and $a_0^{\ 2}+a_2^{\ 2}=1; c_0^{\ 2}+c_2^{\ 2}=1$ (since $\overline{A_1}$ and $\overline{C_1}$ are unit vectors).

By substituting Eqn. (2.3) into Eqn. (2.2), we obtain,

$$\begin{aligned}P_1(t,\theta)=P_0&+(v_0+a_0t+c_0R_1\sin\theta)\overline{A_0^*}\\ &+(v_1+R_1\cos\theta)\overline{A_1^*}\\ &+(v_2+a_2t+c_2R_1\sin\theta)\overline{A_2^*}\end{aligned}$$

That's to say, the coordinates of the cylinder S_1 with regard to the LCS are

$$\begin{cases}x(\theta,t)=v_0+a_0t+c_0R_1\sin\theta\\ y(\theta,t)=v_1+R_1\cos\theta\\ z(\theta,t)=v_2+a_2t+c_2R_1\sin\theta\end{cases}$$

(2.4)

By substituting Eqn. (2.4) into Eqn. (2.1), we obtain the following quartic equation,

$$t^4+a(\theta)t^3+b(\theta)t^2+c(\theta)t+d(\theta)=0$$

(2.5)

with,

$$\mathrm{a}(\theta)=4[\mathrm{a}_0\mathrm{x}(\theta,0)+\mathrm{a}_2\mathrm{z}(\theta,0)]$$

$$\mathrm{b}(\theta)=-4\mathrm{a}_2^{\ 2}\mathrm{R}_0^{\ 2}+2g(\theta)+4[\mathrm{a}_0\mathrm{x}(\theta,0)+\mathrm{a}_2\mathrm{z}(\theta,0)]^2$$

$\mathrm{c}(\theta) = -8\mathrm{a}_2\mathrm{R}_0{}^2\mathrm{z}(\theta,0) + 4[\mathrm{a}_0\mathrm{x}(\theta,0)+\mathrm{a}_2\mathrm{z}(\theta,0)]\cdot \mathrm{g}(\theta)$

$\mathrm{d}(\theta) = g^2(\theta) - 4\mathrm{R}_0{}^2\{[\mathrm{y}(\theta,0)]^2+[\mathrm{z}(\theta,0)]^2\}$

where,

$g(\theta) = -\mathrm{r}_0{}^2+\mathrm{R}_0{}^2+[\mathrm{x}(\theta,0)]^2+[\mathrm{y}(\theta,0)]^2+[\mathrm{z}(\theta,0)]^2$

The approach to finding the roots of the quartic equation $t^4 + at^3 + bt^2 + ct + d = 0$ is presented in the appendix[11,12].

Let θ increase from 0 to 2π by an even step value $\Delta\theta$. For a fixed θ, the cylinder S_1 is parametrized as a line, and the problem of computing intersections between torus and cylinder has been converted to the line/torus intersection problem. Eqn. (2.5) must have four roots t_0, t_1, t_3, t_4. Because the quartic equation has four real roots at most, and because the number of real roots is equal to the number of intersection points between the torus S_0 and the line with respect to θ, we know there are four intersection points at most between the torus S_0 and the line. By discarding imaginary root, and substituting the real roots of the quartic equation for θ into Eqn. (2.4), we can obtain discretized intersection points.

In case that suppose $\overline{A_0}$ is parallel to $\overline{A_1}$ but not collinear, let $\vec{V} = P_1 - P_0$, the projection of P_0 onto the axis of the cylinder $P_0^{'} = P_0 + (\vec{V} - (\vec{V}\cdot\overline{A_1})\cdot\overline{A_1})$, $d = |\overrightarrow{PP_0^{'}}|$, and $\overline{D} = \dfrac{P_0^{'} - P_0}{d}$. We can set the LCS $\{P_0; \overline{A_0^*}, \overline{A_1^*}, \overline{A_2^*}\}$, where, $\overline{A_0^*} = \overline{A_0}$, $\overline{A_1^*} = \overline{D}$, and $\overline{A_2^*} = \overline{A_0^*} \times \overline{A_1^*}$.

In the LCS,

$S_0 : (x^2 + y^2 + z^2 + R_0{}^2 - r_0{}^2)^2 = 4R_0{}^2(y^2 + z^2)$ (2.6)

$S_1 : (y-d)^2 + z^2 = R_1^2$

(2.7)

From Eqn. (2.7) we have,

$z^2(y) = R_1^2 - (y-d)^2$ ($|y-d| \le R_1$)

(2.8)

and $y^2 + z^2 = R_1^2 + 2y\cdot d - d^2$

(2.9)

By substituting Eqn. (2.9) into Eqn. (2.6), we can obtain

$$x^2(y) = r_0^2 - R_0^2 - (R_1^2 + 2y\cdot d - d^2) \pm 2R_0\sqrt{R_1^2 + 2y\cdot d - d^2}$$

Therefore, the intersection curves can be expressed as follows:

$$\begin{cases} x^2 = x^2(y) = r_0^2 - R_0^2 - (R_1^2 + 2y\cdot d - d^2) \\ \pm 2R_0\sqrt{R_1^2 + 2y\cdot d - d^2} \\ y = y \\ z^2 = z^2(y) = R_1^2 - (y-d)^2 \end{cases}$$

and the domain of y must satisfy $x^2(y) \ge 0$ and $z^2(y) \ge 0$.

Finally, for the case that $\overline{A_0}$ and $\overline{A_1}$ are collinear, set the LCS $\{P_0; \overline{A_0^*}, \overline{A_1^*}, \overline{A_2^*}\}$, where, $\overline{A_0^*} = \overline{A_0}$, $\overline{A_1^*}$ is a unit vector which is perpendicular to $\overline{A_0^*}$, and $\overline{A_2^*} = \overline{A_0^*} \times \overline{A_1^*}$.

In this coordinate system,

$S_0 : (x^2 + y^2 + z^2 + R_0{}^2 - r_0{}^2)^2 = 4R_0{}^2(y^2 + z^2)$ (2.10)

$S_1 : y^2 + z^2 = R_1^2$

(2.11)

By substituting Eqn. (2.11) into Eqn. (2.10), we can obtain,

$x^2 = r_0^2 - (R_0 - R_1)^2$ (since $r_0 \le R_0$)

Hence, the intersection curves can be explicitly expressed as follows:

$$\begin{cases} x^2 = r_0^2 - (R_0 - R_1)^2 \\ y^2 + z^2 = R_1^2 \end{cases}$$

That's to say, the intersection curves for this case are one profile circle ($|R_0 - R_1| = r_0$) or two profile circles ($|R_0 - R_1| < r_0$) or empty ($|R_0 - R_1| > r_0$).

Kim[8] have prensented algebraic conditions under which torus/cylinder intersections are conic sections, we present here geometric and intuitive ways to identify and compute the conic sections in the following.

(a) If the two axes are collinear, the intersections are one profile circle or two profile circles or empty.

(b) The point Q is the intersection point of the plane $\{P_0; \overline{A_1}\}$ and the axis $\{P_1; \overline{A_1}\}$.

If $\overline{A_0} \perp \overline{A_1}$, $\overline{P_0Q} \perp \overline{A_0}$, $|\overline{P_0Q}| = R_0$, and $R_1 = r_0$, then one of the intersection curves is a cross-sectional circle, which is on the plane $\{Q; \overline{A_1}\}$, with Q as the center, and $R_1 = r_0$ as the radius, as shown in Fig. 2 (a).

(c) Suppose the plane $\{P_0;\overline{A_0}\}$ and the axis $\{P_1;\overline{A_1}\}$ has an intersection Q.

If $\overline{P_0Q} \perp \overline{A_1}$, $|\overline{P_0Q}| = r_0$, $R_1 = R_0$, and $(\frac{\overline{A_0}\cdot\overline{A_1}}{|\overline{A_0}|\cdot|\overline{A_1}|})^2 = 1 - \frac{r_0^2}{R_0^2}$, then one of the intersection curves is an Yvone-Villiaroeau circle[7], which is on the plane $\{Q;\overline{A_1}\}$, with Q as the center, and $R_1 = R_0$ as the radius, as shown in Fig. 2. (b).

2.2 Torus and Cone Intersections

The algorithm for computing torus/cone intersections is similar to the algorithm for computing torus/ cylinder intersections.

Suppose that torus S_0 has a center P_0, axis direction $\overline{A_0}$ ($|\overline{A_0}| = 1$), major radius R_0, and minor radius r_0. Suppose that cone S_1 has an apex P_1, positive axis direction $\overline{A_1}$ ($|\overline{A_1}| = 1$), and half angel β.

If $\overline{A_0}$ isn't parallel to $\overline{A_1}$

$$S_0 : (x^2 + y^2 + z^2 + R_0^2 - r_0^2)^2 = 4R_0^2(y^2 + z^2)$$

$$\begin{aligned} S_1 : P_1(t,\theta) = {} & P_1 + t\overline{A_1} \\ & + (t\cdot\tan\beta)\cos\theta\overline{B_1} \\ & + (t\cdot\tan\beta)\sin\theta\overline{C_1} \end{aligned}$$

where,

$$\begin{cases} \overline{A_1} = a_0\overline{A_0^*} + a_1\overline{A_1^*} + a_2\overline{A_2^*} = a_0\overline{A_0^*} + a_2\overline{A_2^*} \\ \overline{B_1} = \overline{A_1^*} \\ \overline{C_1} = c_0\overline{A_0^*} + c_1\overline{A_1^*} + c_2\overline{A_2^*} = c_0\overline{A_0^*} + c_2\overline{A_2^*} \end{cases}$$

Hence,

$$\begin{aligned} P_1(t,\theta) = {} & P_0 + (v_0 + a_0 t + c_0 t\cdot\tan\beta\sin\theta)\overline{A_0^*} \\ & + (v_1 + t\cdot\tan\beta\cos\theta)\overline{A_1^*} \\ & + (v_2 + a_2 t + c_2 t\cdot\tan\beta\sin\theta)\overline{A_2^*} \end{aligned}$$

That's to say, the coordinates of the cone S_1 with regard to the LCS are

$$\begin{cases} x(\theta,t) = v_0 + a_0 t + c_0 t\cdot\tan\beta\sin\theta \\ y(\theta,t) = v_1 + t\cdot\tan\beta\cos\theta \\ z(\theta,t) = v_2 + a_2 t + c_2 t\cdot\tan\beta\sin\theta \end{cases}$$

The rest of the analysis is the same as the torus/cylinder intersections, and we omit it here.

Similarly,

(a) If the two axes are collinear, the intersections are one profile circle or two profile circles or empty.

(b) The point Q is the intersection point of the plane $\{P_0;\overline{A_1}\}$ and the radial $\{P_1;\overline{A_1}\}$.

If $\overline{A_0} \perp \overline{A_1}$, $\overline{P_0Q} \perp \overline{A_0}$, $|\overline{P_0Q}| = R_0$, and $|\overline{P_1Q}|\cdot\tan\beta = r_0$, then one of the intersection curves is a cross-sectional circle, which is on the plane $\{Q;\overline{A_1}\}$, with Q as the center, and r_0 as the radius.

(c) Suppose the plane $\{P_0;\overline{A_0}\}$ and the radial $\{P_1;\overline{A_1}\}$ has an intersection Q.

If $\overline{P_0Q} \perp \overline{A_1}$, $|\overline{P_0Q}| = r_0$, $|\overline{P_1Q}|\cdot\tan\beta = R_0$, and $(\frac{\overline{A_0}\cdot\overline{A_1}}{|\overline{A_0}|\cdot|\overline{A_1}|})^2 = 1 - \frac{r_0^2}{R_0^2}$, then one of the intersection curves is an Yvone-Villiaroeau circle[7], which is on the plane $\{Q;\overline{A_1}\}$, with Q as the center, and R_0 as the radius.

3. SORTING DISCRETE INTERSECTION POINTS

After obtaining the unordered intersection points, we must sort them properly to form intersection curves. In this paper, we sort the intersection points based on the line parameter incremental used in solving the governing Eqn. (2.5) and on the spatial adjacency of the intersection points.

The roots of the quartic equation $t^4 + at^3 + bt^2 + ct + d = 0$ can be classified as the following three cases:

4 real roots: in this case, four intersection points. They are sorted in the parameter decreasing order, $t_0 \ge t_1 \ge t_2 \ge t_3$.

2 real roots and 2 imaginary roots: in this case, two intersection points. We sort them in the parameter decreasing order, $t_0 \ge t_1$.

no real roots: in this case, no intersection points.

3.1 Sorting the Intersection Points

Preprocessing: Traversal through all the intersection points, and find an angle θ^* such that Eqn. (2.5) has the most roots. There are possible four, two or zero real roots. If the maximum number of roots is zero, then the torus doesn't intersect the cylinder or the cone, and no further considerations are needed. Otherwise, we sort them in the decreasing order $t_0[0] \geq ... \geq t_0[n-1]$, $n \in \{2,4\}$. Denote the intersection points as $P_0[0], ..., P_0[n-1]$.

The Sorting Algorithm: Starting with $\theta = \theta^*$, get the intersection points with respect to $\theta + \Delta\theta$ with parameter values $t_1[0] \geq ... \geq t_1[m-1]$. By doing this, we obtain two sets of ordered intersection points: $P_0[0], ..., P_0[n-1]$ for $t_0[0] \geq ... \geq t_0[n-1]$, and $P_1[0], ..., P_1[m-1]$ for $t_1[0] \geq ... \geq t_1[m-1]$, with $n,m \in \{2,4\}$. Variable $t_0[i]$ corresponds to point $P_0[i]$, variable $t_1[j]$ corresponds to point $P_1[j]$.

In case that m=0, the intersection curves all end at $P_0[0], ..., P_0[n-1]$, and we incrementally increase angle θ with the step value $\Delta\theta$ until the intersections occur or no intersections found anymore. In the former case, these intersection points are the starting points of new intersection curves. In the later case, the sorting process ends. Repeat Step 2 to get two sets of intersection points.

(1) If n=m, corresponding intersection points $P_0[k]$, $P_1[k]$ ($k \in \{0,1,...,n-1\}$) on the two straight line are connected by straight lines.

(2) else if n > m, then n=4, m=2, we compute the sum of the differences between two variables as follows:

$|t_0[0]-t_1[0]|+|t_0[1]-t_1[1]|=\delta_{01}$,
$|t_0[0]-t_1[0]|+|t_0[2]-t_1[1]|=\delta_{02}$,
$|t_0[0]-t_1[0]|+|t_0[3]-t_1[1]|=\delta_{03}$,
$|t_0[1]-t_1[0]|+|t_0[2]-t_1[1]|=\delta_{12}$,
$|t_0[1]-t_1[0]|+|t_0[3]-t_1[1]|=\delta_{13}$,
$|t_0[2]-t_1[0]|+|t_0[3]-t_1[1]|=\delta_{23}$

Suppose δ_{ij} is the minimum of $\delta_{01}, \delta_{02}, \delta_{03}, \delta_{12}, \delta_{13}, \delta_{23}$, then $t_0[i]$ corresponds to $t_1[0]$, $t_0[j]$ corresponds to $t_1[1]$, and hence we connect $P_0[i]$ with $P_1[0]$, and $P_0[j]$ with $P_1[1]$.

There remains two points that are not connected to any other point during this process. They are the ending points of the intersections. Moreover, if there are more than one δ-s that are equal to the minimum value, the one with minimal angular span is selected.

(3) else (n=2, m=4) Similar to the case of n=4, m=2, we compute the sum of the differences between two variables as follows:

$|t_0[0]-t_1[0]|+|t_0[1]-t_1[1]|=\mu_{01}$,
$|t_0[0]-t_1[0]|+|t_0[1]-t_1[2]|=\mu_{02}$,
$|t_0[0]-t_1[0]|+|t_0[1]-t_1[3]|=\mu_{03}$,
$|t_0[0]-t_1[1]|+|t_0[1]-t_1[2]|=\mu_{12}$,
$|t_0[0]-t_1[1]|+|t_0[1]-t_1[3]|=\mu_{13}$,
$|t_0[0]-t_1[2]|+|t_0[1]-t_1[3]|=\mu_{23}$

Suppose μ_{ij} is the minimum of $\mu_{01}, \mu_{02}, \mu_{03}, \mu_{12}, \mu_{13}, \mu_{23}$, then $t_0[0]$ corresponds to $t_1[i]$, $t_0[1]$ corresponds to $t_1[j]$, and hence we connect $P_0[0]$ with $P_1[i]$, and $P_0[1]$ with $P_1[j]$.

There remains two points that are not connected to any other point during this process. They are the starting points of the intersections. Moreover, if there are more than one μ-s that are equal to the minimum value, the one with minimal angular span is selected.

Repeat $\theta \leftarrow \theta + \Delta\theta$ Until θ reaches 2π.

Postprocessing: We can obtain at most eight polylines after we finish Step 2. Sometimes, the intersection gap between two polylines lies within a parameter threshold ε, and we need to refine the polylines as follows:

(1) For the first/last (last/first) point (θ_{i0}, t, P) on a polyline and the first/last point (θ_{j0}, t', Q) on another, if $\theta_{j0} = \theta_{i0} \pm \Delta\theta$ and $|t - t'| < \varepsilon$, we connect P and Q, and combine the two polylines.

(2) For the first/last (last/first) point (θ_{i0}, t, P) on a polyline and the first/last point (θ_{j0}, t', Q) on another, if $\theta_{j0} = \theta_{i0}$ and $|t - t'| < \varepsilon$, we connect P and Q, and combine the two polylines.

(3) If there is a polyline can be combined with more than one polylines, we select a polyline with the minimal angular span to combine with. This is done only for intersection curves that go cross themselves.

If torus and cylinder or cone intersection curves are regular (the intersection curves don't go cross themselves), we needn't consider condition (3) of the above Postprocessing procedure when we combine these polylines, as shown in Fig. 1. On the other hand, if the intersection curves are irregular, we must consider the condition, as shown in Fig. 2 and Fig. 3. Moreover, the cases that there are conic section intersections are illustrated in Fig. 2.

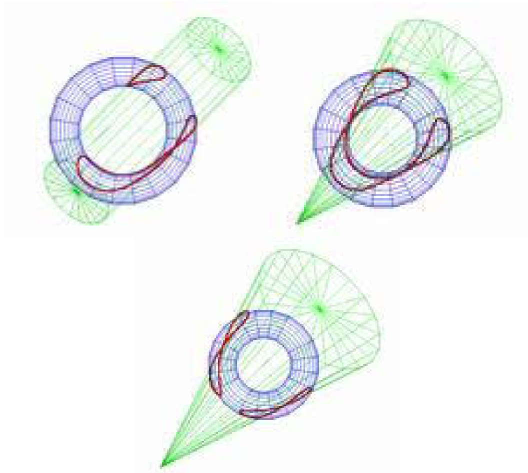
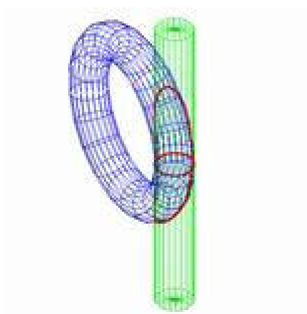


Fig. 1. Examples of regular intersection curves



(a) One Cross-Sectional Circle and one non-planar section

(b) One Yvone-Villiaroeau Circle and one non-planar section

Fig. 2. Examples of one conic section and one non-planar intersection

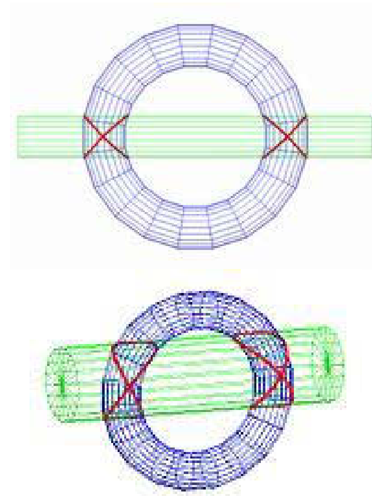


Fig. 3. Examples of two irregular intersection curves

4. TORUS AND SPHERE INTERSECTIONS

The intersection between a torus and a sphere can be expressed explicitly by algebraic formulae. Suppose that torus S_0 has a center P_0, axis direction $\overline{A_0}$ ($|\overline{A_0}|=1$), major radius R_0, and minor radius r_0. Suppose that sphere S_1 has a center P_1, and radius R_1.

Let, $\overline{V} = P_1 - P_0$,

$$dx = \begin{cases} 0, & if\ |\overline{V}|=0 \\ \overline{A_0}\cdot\overline{V}, & otherwise \end{cases} \quad , \quad \text{and}$$

$$dz = \begin{cases} 0, & if\ |\overline{V}|=0 \\ -|\overline{V}|\cdot\sqrt{1-\cos^2\varphi} \le 0, & otherwise \end{cases}$$

where, $\cos\varphi = \dfrac{\overline{A_0}\cdot\overline{V}}{|\overline{V}|}$

We can set up the LCS as follows:

(1) if $|\overline{V}|=0$ or $|\cos\varphi|=1$, then the LCS is set up as $\{P_0;\overline{A_0^*},\overline{A_1^*},\overline{A_2^*}\}$, where, $\overline{A_0^*}=\overline{A_0}$, $\overline{A_1^*}$ is a unit vector which is perpendicular to $\overline{A_0^*}$, and $\overline{A_2^*}=\overline{A_0^*}\times\overline{A_1^*}$.

(2) else, let $\overline{A_1^*} = \dfrac{\overline{A_0}\times\overline{V}}{|\overline{A_0}\times\overline{V}|}$, $\overline{A_2^*}=\overline{A_0^*}\times\overline{A_1^*}$, and take the LCS as $\{P_0;\overline{A_0^*},\overline{A_1^*},\overline{A_2^*}\}$.

In the LCS, the torus and the sphere become

$$S_0 : (x^2+y^2+z^2+R_0{}^2-r_0{}^2)^2 = 4R_0{}^2(y^2+z^2) \tag{4.1}$$

$S_1 : (x-dx)^2 + y^2 + (z-dz)^2 = R_1^2$
(4.2)

From Eqn. (4.2) we have

$$x^2 + y^2 + z^2 = R_1^2 - (dx)^2 + 2x\cdot(dx) - (dz)^2 + 2z\cdot(dz) \quad (4.3)$$

and

$$y^2 + z^2 = R_1^2 - (dx)^2 + 2x\cdot(dx) - (dz)^2 + 2z\cdot(dz) - x^2 \quad (4.4)$$

By substituting Eqn. (4.3) and Eqn. (4.4) into the left and the right sides of Eqn. (4.1) respectively, we can obtain the following quadratic equation about variable x (or variable z).

$$a(z)\cdot x^2 + b(z)\cdot x + c(z) = 0$$

with

$$a(z) = 4\cdot(dx)^2 + 4R_0^2$$

$$b(z) = -8R_0^2\cdot(dx) + 4(dx)\cdot f(z)$$

$$c(z) = -4R_0^2\cdot[-(dx)^2 - (dz)^2 + R_1^2 + 2z\cdot(dz)] + f^2(z)$$

where,

$$f(z) = -(dx)^2 - (dz)^2 - r_0^2 + R_0^2 + R_1^2 + 2z\cdot(dz)$$

Therefore, the intersection can be expressed as follows:

$$\begin{cases} x = x_0(z) = [-b(z) + \sqrt{\Delta}]/[2a(z)] \\ y^2(z) = R_1^2 - [x_0(z) - dx]^2 - (z-dz)^2 \\ z = z \end{cases}$$

$$\begin{cases} x = x_0(z) = [-b(z) - \sqrt{\Delta}]/[2a(z)] \\ y^2(z) = R_1^2 - [x_0(z) - dx]^2 - (z-dz)^2 \\ z = z \end{cases}$$

with $\Delta = b^2(z) - 4a(z)\cdot c(z)$,

where, $-R_0 - r_0 \le z \le R_0 + r_0$, and the domain of z must satisfy $y^2(z) \ge 0$, $\Delta \ge 0$.

Fig. 4 illustrates two general examples of torus/sphere intersections.

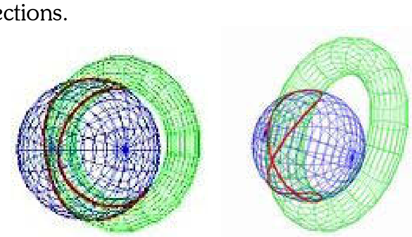


Fig. 4. Two examples of Torus/Sphere Intersections

Kim[8] have presented algebraic conditions under which torus/sphere intersections are conic sections, we present here geometric and intuitive ways to identify and compute the conic sections in the following.

(a) If $dz = 0$, the intersection curves are one profile circle or two profile circles or empty, as shown in Fig. 5 (a).

(b) In the LCS,if $dx = 0$, $dz = -R_0$, and $R_1 = r_0$ (i.e. $\overline{P_0P_1} \perp \overline{A_0}$, $|\overline{P_0P_1}| = R_0$ and $R_1 = r_0$), then the intersection curve is a cross-sectional circle, which is on the plane $\{P_1; \overline{A_0} \times \overline{P_0P_1}\}$, with P_1 as the center, and $R_1 = r_0$ as the radius, as shown in Fig. 5 (b).

(c) In the LCS, let $Q_0\{0, \frac{R_0\cdot\sqrt{R_1^2 - r_0^2}}{dz}, \frac{R_0^2}{dz}\}$, $Q_1\{0, -\frac{R_0\cdot\sqrt{R_1^2 - r_0^2}}{dz}, \frac{R_0^2}{dz}\}$.

If $dx = 0$, $R_1 > r_0$ and $dz = -\sqrt{(R_1^2 - r_0^2) + R_0^2}$ (i.e. $\overline{P_0P_1} \perp \overline{A_0}$, $R_1 > r_0$ and $|\overline{P_0P_1}| = \sqrt{(R_1^2 - r_0^2) + R_0^2}$), then the intersection curves are composed of two cross-sectional circles. One is on the plane $\{Q_0; \overline{Q_0P_1}\}$, with Q_0 as the center, and r_0 as the radius; the other is on the plane $\{Q_1; \overline{Q_1P_1}\}$, with Q_1 as the center, and r_0 as the radius, as shown in Fig. 5 (c).

(d) In the LCS,

① let, $x' = 0$, $y'_{0,1} = \pm\sqrt{r_0^2 - \frac{r_0^4}{(dz)^2}}$, $z' = \frac{r_0^2}{dz}$, $Q_0\{x', y'_0, z'\}$, $Q_1\{x', y'_1, z'\}$.

If ($R_0^2 + (dx - x')^2 + (0 - y_0')^2 + (dz - z')^2 = R_1^2$ and $(\frac{\overline{A_0}\cdot\overline{Q_0P_1}}{|\overline{A_0}|\cdot|\overline{Q_0P_1}|})^2 = 1 - \frac{r_0^2}{R_0^2}$)

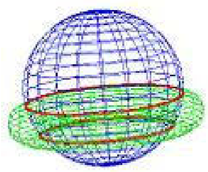
or

($R_0^2 + (dx - x')^2 + (0 - y_1')^2 + (dz - z')^2 = R_1^2$ and $(\frac{\overline{A_0}\cdot\overline{Q_1P_1}}{|\overline{A_0}|\cdot|\overline{Q_1P_1}|})^2 = 1 - \frac{r_0^2}{R_0^2}$) ,

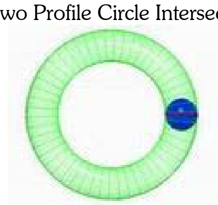
then the intersections are composed of two Yvone-Villiaroeau circles[7]. One is on the plane $\{Q_0; \overline{Q_0P_1}\}$, with Q_0 as the center, and R_0 as the radius; the other is on the plane $\{Q_1; \overline{Q_1P_1}\}$, Q_1 is the center, and R_0 as the radius, as shown in Fig. 5 (d).

② let, $x'_{0,1} = \pm\sqrt{R_0^{\ 2} - r_0^{\ 2} - (\frac{r_0^{\ 2} - R_0^{\ 2}}{R_0})^2}$, $y' = \frac{r_0^{\ 2} - R_0^{\ 2}}{R_0}$, $z' = 0$, $Q_0\{x'_0, y', z'\}$, $Q_1\{x'_1, y', z'\}$, $Q\{0, -R_0, 0\}$, $P\{0, 0, -r_0\}$.

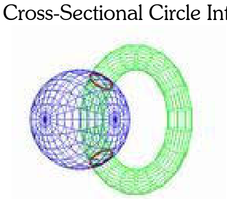
If $dx = 0$, $dz = -r_0$ and $R_0 = R_1$ (where, $dy \equiv 0$), then the intersections are composed of two Yvone-Villiaroeau circles[7]. One is on the plane $\{P; \overline{QQ_1}\}$, with P as the center, and R_0 as the radius; the other is on the plane $\{P; \overline{QQ_1}\}$, with P as the center, and R_0 as the radius, as shown in Fig. 5 (*e*).



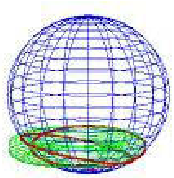
(a) Two Profile Circle Intersections



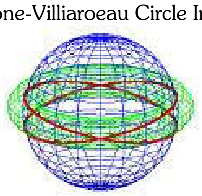
(b) One Cross-Sectional Circle Intersection



(c) Two Cross-Sectional Circle Intersections



(d) Two Yvone-Villiaroeau Circle Intersections



(e) Two Yvone-Villiaroeau Circle Intersections

Fig. 5. Conic Intersections between Torus and Sphere

5. CONCLUSION

Natural quadratic surfaces (cylinder, cone and sphere) and torus are most widely used surfaces in CAD systems. The efficiency and robustness of computing intersection curves between these types of surfaces are critical to the performance and robustness of the CAD systems. In this paper, we present algebraic algorithms for computing intersection curves between torus and cylinder, cone and sphere. For torus/cylinder and torus/cone intersections, we present efficient and robust algorithms based on the governing quartic equation with line generator parametrization of the cylinder or cone. By solving the governing quartic equation, we obtain discrete intersection points. We then present an algorithm for sorting the intersection points based on the line parameter incremental used in solving the governing equation and on the spatial adjacency of the intersection points. For torus/sphere intersections, we are able to explicitly express the intersection curves in analytical forms. Furthermore, we present special cases with intuitive geometric configurations under which the conic intersection curves exist. Future work includes algebraic approaches for torus/torus intersections.

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APPENDIX: Finding the Roots Using Ferrari's Formula[11,12]

A quartic equation,

$$t^4 + a_1 t^3 + a_2 t^2 + a_3 t + a_4 = 0 \tag{A1}$$

has the resolvent cubic equation.

$$y^3 + b_1 y^2 + b_2 y + b_3 = 0 \tag{A2}$$

where,

$$b_1 = -a_2$$

$$b_2 = a_1 a_3 - 4a_4$$

$$b_3 = 4a_2 a_4 - a_3^{\,2} - a_1^{\,2} a_4$$

Let y_1 be the real root of Eqn. (A2), then the four roots of Eqn. (A1) are obtained from the roots of the following quadric equation (when the Eqn. (A2) has two real roots, either choice of y_1 will lead to the same result.) [11,12]:

$$z^2 + c_1 z + c_2 = 0$$

where,

$$c_1 = (a_1 \pm \sqrt{a_1^{\,2} - 4a_2 + 4y_1})/2$$

$$c_2 = (y_1^{\,2} \mp \sqrt{y_1 - 4a_4})/2$$