Interpolating Multiple Intersecting Curves Using Catmull-Clark Subdivision Surfaces

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ABSTRACT

The problem of constructing a smooth subdivision surface interpolating multiple intersecting curves was partially addressed in the literature. In the context of Doo-Sabin subdivision surfaces, Nasri[3] presented a solution to interpolate unlimited number of curves through an extraordinary point. In the Catmull-Clark setting, no more than two intersecting curves could so far be interpolated. That is, the interpolation of multiple intersecting curves remains a non-trivial and elusive problem. This paper puts forth a solution to this problem. The solution relies in a fundamental way on the by-now well-known notion of Catmull-Clark Polygonal Complexes introduced in [5].

Keywords: Catmull-Clark Subdivision Surfaces, Polygonal Complexes, Curve Interpolation.

1. INTRODUCTION

Several authors contributed to the interpolation of curves by subdivision surfaces. Initiated by Nasri back in 1995 [3], an approach was described to generate a smooth Doo-Sabin subdivision surface through a mesh of curves. The approach was extended in 1997 [4] to the interpolation of dangling quadratic B-spline curves. The control polygon of such a curve can be obtained by tagging edges and vertices on the polyhedron describing the subdivision surface. The first attempt to study this interpolation problem in the Catmull-Clark setting was made by Schweitzer in 1997 [9] and later by Levin [2], who proposed for that the Combined Subdivision Scheme in 1999. The general use of polygonal complexes to interpolate curves by subdivision surfaces was proposed by Nasri in [5]. These complexes provide a general framework for the interpolation of intersecting or dangling curves under any subdivision scheme. Since the limit of subdivision of a polygonal complex is a curve, the key idea is to identify a polygonal complex within the control mesh representing the surface. This simply means that the limit curve will very naturally be interpolated by the limit surface of the corresponding control mesh. Polygonal complexes were initially proposed for the Doo-Sabin subdivision scheme and later for the Catmull-Clark scheme [6].

As it currently stands, the problem of interpolating intersecting curves by a smooth surface is solved under the constraint that no more than two curves can intersect at any given point. This problem was recently addressed by Nasri [8], where an approach was proposed for the interpolation of unlimited number of curves through an extraordinary point in the context of Doo-Sabin subdivision surfaces. Nevertheless, this remained a challenging problem in the context of Catmull-Clark subdivision surfaces.

This paper presents an intuitively-clear and easy to implement technique to solve this exact problem. The particular approach being followed here is similar to what is outline above. In fact, the idea is to have curve information embodied as limits of subdivision of polygonal structures designed as an integral part of the control mesh representing the surface. As a result, the curves limit of these structures will very naturally reside on the limit subdivision surface of the initial control mesh.

Our previous results [6, 7] on the interpolation of curves by a Catmull-Clark (CC) subdivision surface are constrained by factors that are worth noting. In fact, each curve to be interpolated has to be manipulated in isolation (i.e. it cannot intersect with any other of the given curves) and either it has to be closed or, when it is open, the surface has to be open and the curve has to stretch from one border vertex of the surface to another border vertex.

These limitations will surely need to be overcome before a theory of the subject can be said to be complete. As

such, the treatment of intersecting curves in this setting has to be seen as an effort in this direction.
This paper proceeds as follows. Section 2 reminds the reader of the basics of the Catmull-Clark subdivision scheme. Section 3 contains a review of the notion of CC polygonal complexes and its role in achieving the interpolation of individual curves by CC subdivision surfaces. Section 4 introduces the notion of X-Configurations and section 5 identifies its role in achieving the interpolation of multiple intersecting curves by CC subdivision surfaces. Section 6 identifies some useful directions leading from the research ideas reported in this paper. The final sections of the paper conclude with a general summary and some useful conclusions.

2. CC SUBDIVISION AND LIMIT SURFACES

In a CC subdivision step [1], a control mesh M is subdivided into another control mesh M', as follows (see Figure 1).

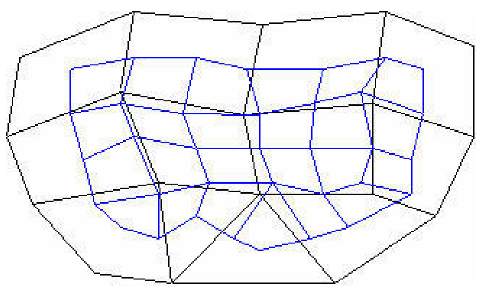


Fig. 1. The CC Subdivision Scheme

Each inner face F of the mesh gives rise to an F-vertex that is the average of the vertices of the face F. Each inner edge E gives rise to an E-vertex that is the average of the vertices of the edge together with the F-vertices of the adjacent faces of E. Each inner vertex v gives rise to a V-vertex as specified by the following formula:

$$((n-2)*v + (R + S)/n)/n$$

where

- n is the number of faces adjacent to v
- R = Sum ({v_i; i = 1 .. n}), where vv_i is an edge of the corresponding mesh.
- S = Sum ({v_{fi}; i = 1 .. n}), where v_{fi} is an F-vertex of a face f_i of the mesh adjacent to the vertex v.

At the end of this process, each F-vertex is connected to the adjacent E-vertices and each E-vertex is connected to the adjacent V-vertices. The resulting faces will form the new subdivided mesh. In this context, note that repeated application of this subdivision process will in general lead to more faces and smaller edges. At the limit, this will converge to a smooth surface.
According to this formulation, the border edges and vertices do not contribute any new vertices. Therefore, these vertices and edges are generally kept away from the limit surface. However, these vertices and edges are sometimes incorporated into the main subdivision routine as special cases.

3. CC POLYGONAL COMPLEXES

A *simple* CC polygonal complex is a 3×n matrix M of points representing three *parallel* polygons (t_i), (m_i) and (b_i), all of the same length. These may be seen as a sequence of pairs of rectangular faces, where each pair of faces of this sequence has a common edge and each two consecutive pairs have common respective edges.

A *general* CC polygonal complex is encountered when the polygons (t_i), (m_i) and (b_i) do not all have the same length. That is, the corresponding faces are not all rectangular at the outer edges. However, it is important to note here that each inner vertex of a CC complex must be regular in the sense that it connects exactly four edges.

Note here that a general CC complex can be transformed into a simple one after be performing a single CC subdivision step. Note also that a CC complex is interesting because, under subdivision, it gets transformed into a thinner and thinner complex which, at the limit, converges to a smooth curve.
In this context, the limit of a simple CC complex M is a B-spline curve whose control polygon P is given by the following formula (see [6]):

$$(1/6)*[1\ 4\ 1]*M \qquad (1)$$

The direct use of that is as follows. When a complex is embodied within a control mesh, its limit curve will then naturally be interpolated by the limit surface of this mesh

Likewise, if a CC complex M is subdivided one step into M' and if the limit polygon P of M is also subdivided one step into P', then the property stated in equation(1) will be preserved under subdivision. That is, P' will be equal to (1/6)*[1 4 1]*M'. This observation is worth noting as it is quite crucial to the analysis conducted in this paper.

Now, if a complex M' is obtained from a CC complex M by substituting the mid-polygon m of M by the polygon (see [7]):

$$m'=(1/4)*[-1\ 6\ -1]*M \qquad (2)$$

then the limit of M' is a B-spline curve identical to that of m.

The direct use of that is as follows. Given a curve defined by a control polygon (m_i), we can turn it into a polygonal complex M by adding to it two more rows of points (t_i) and (b_i). Applying the transformation (2), we can guarantee that any mesh embodying the complex M' will in fact be interpolating the original curve defined by (m_i).

4. X-CONFIGURATIONS

We have established in the previous section the correspondence between CC complexes and B-spline curves and the usefulness of that for interpolating isolated (non-intersecting) curves. In this section, we will see how this notion can be extended to deal with situations where the given curves can intersect.

It is to be noted that if two complexes meet end-to-end, their limit curves do meet but are generally different from their original versions in the immediate neighborhood of their meeting point (see Figure 2).

This is to be expected, since the middle vertex there is no longer a border vertex. However, this vertex will be regular. The same observation is true in the case where the number of complexes is four (see Figure 3).

In both cases, *it would be interesting to determine what exactly happens at the meeting point of these complexes.*

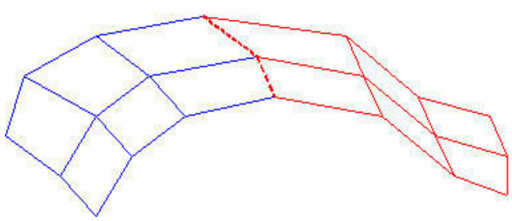


Fig. 2. Intersection of Two Complexes

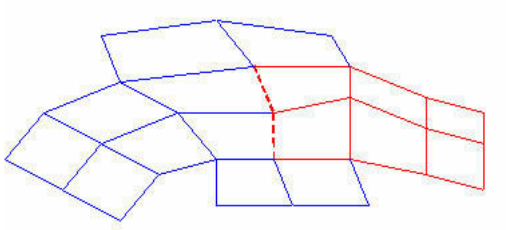


Fig. 3. Intersection of Four Complexes

In the situation where the number of complexes is not two or four, the center vertex will not be a regular vertex. Thus, each connecting complex will not be regular around the center vertex in the sense discussed above, because all polygons of various complexes will have this vertex as their meeting point.

Definition 1: an X-Slice is a closed polygon with one of its vertices marked as its starting point.
Definition 2: an X-Configuration is composed of an *even* number n ($n \geq 4$) of X-Slices, all adjacent (one to the next and the last to the first) around the same starting point.

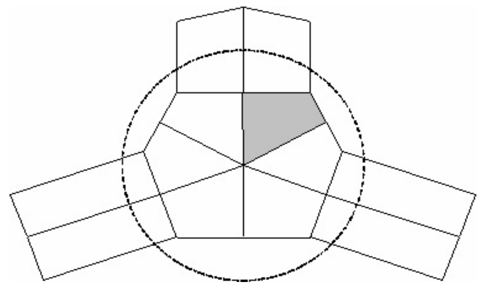


Fig. 4. An X-Configuration, the shaded face is an X-Slice of this configuration

It is easy to see that a one-step subdivision of an X-Configuration will result in an X-Configuration and the limit of subdivision of an X-Configuration is a point corresponding to its innermost vertex.

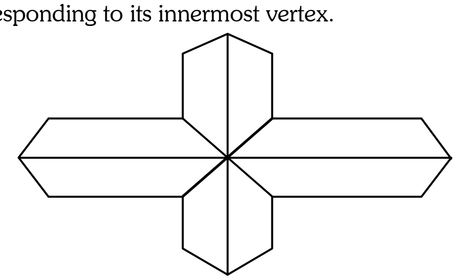


Fig. 5. The Symmetry Condition

Now, a symmetry condition can be formulated in such a way that, if satisfied by an X-Configuration, will leave the centre of the X-Configuration undisturbed under CC subdivision.

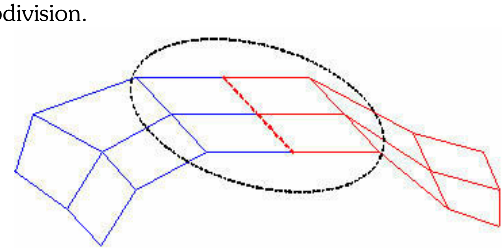


Fig. 6. Intersection at a symmetric X-Configuration

A preliminary version of this condition can be stated as follows (see Figure 5): the 2k X-Slices go around the centre of the X-Configuration in successive pairs. Each of those pairs has its component X-Slices symmetric with respect to their common edges. Furthermore, each X-Slice is symmetric to the one directly opposite to it with respect to the center of the X-Configuration.

Note here that a more general (but more subtle) version of this condition can arguably be formulated. Note also that this symmetry condition does not imply that all the vertices of the X-Configuration involved have to be coplanar (see Figure 11 below for an illustration of that).

Now**,** if any number n (n ≥ 2) of complexes meet at a symmetric planar X-Configuration of the kind discussed above, then one would be justified in expecting that the corresponding limit curves will meet at the centre of this X-Configuration and that the surface will be tangent-plane continuous there.

For instance, if two complexes meet at symmetric X-Configuration, then the curves corresponding to these two complexes will meet at the center of this X-Configuration and will be smooth there (see Figure 6).

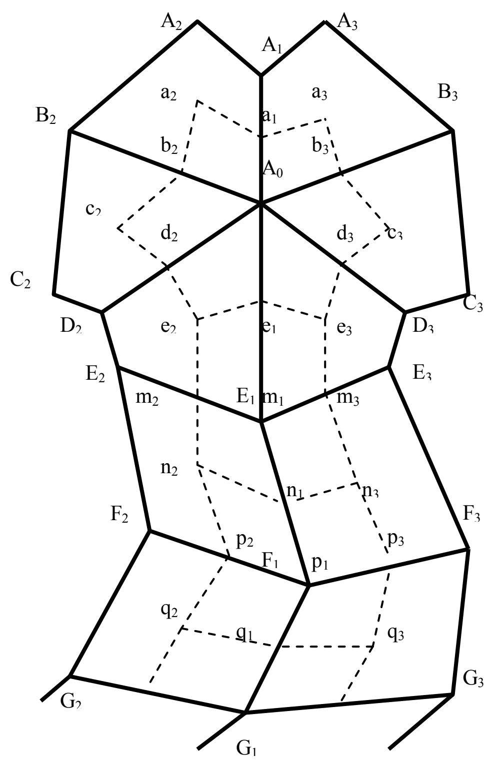


Fig. 7. X-Complex Subdivision

5. INTERSECTION OF CC COMPLEXES

Definition 3: an X-Complex is a group of two or more CC polygonal complexes connected to a common X-Configuration (see an illustration of that in Figure 4 above). Thus, an X-Configuration acts as a docking station where CC complexes may be connected to any of its available ports.

In the irregular situation depicted in Figure 4, the limit of any docking polygonal complex (P) cannot be established via the formula stated in equation (1). Moreover, it is not at all obvious how to construct a CC polygon that, when subdivided in the conventional sense, will satisfy the properties derived from (1) for (P) (see second paragraph following equation (1) above). This is true, specifically in the neighborhood of the center vertex of the X-Configuration (see Appendix, for more details).

5.1 Virtual Faces

In order to handle this specific problem, we will resort to the notion of *virtual* faces.

Definition 4: Given a face F of a mesh, a virtual face corresponding to F is a face that is not actually part of the mesh, but it is used instead of F to derive the F-vertex corresponding to F after a one-step subdivision.

This notion will be used for the purpose of obtaining the desired interpolation effects at the limit of the subdivision process.

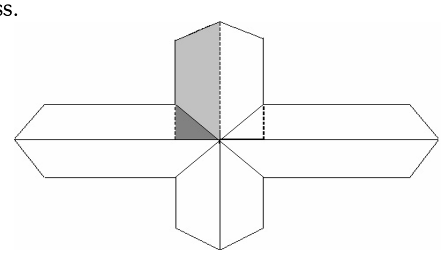


Fig. 8. X-Slices stretching into virtual faces

The way these virtual faces are constructed is as follows: the extremity side of a polygonal complex connected to an X-Shape is stretched toward the centre of the X-Shape (see Figure 8). This is done in such a way as to have this centre as the midpoint of this extremity side (also see construction below).

It is important to note here that these virtual faces are introduced at the start of the subdivision process. These will be subdivided and used to derive alternative F-vertices at every step of the subdivision; that is, until the limit of the subdivision is reached. This in fact amounts to altering the coefficients of the subdivision scheme at and around the center of the irregular X-Configuration. However, these new coefficients will remain constant in those regions throughout the subdivision process.

5.2 Virtual Face Construction

In the construction below, A_0 is the centre of the symmetric X-Configuration (see Figures 7 and 9). The vertices e_2 (resp. e_3) are the F-vertices corresponding to the face $A_0E_1E_2D_2$ (resp. $A_0E_1E_3D_3$). We need to create virtual faces $A_0E_1E_2D'_2$ and $A_0E_1E_3D'_3$ such that

$A_0 = (D'_2 + D'_3)/2$

This way, we obtain the new desired F-vertices e'_2 and e'_3 as required. We do that by extending E_1A_0 in the direction of A_0 and finding a point X there so that ($0 \le u \le 1$):

$D'_2 = uX + (1-u)D_2$
$D'_3 = uX + (1-u)D_3$

This will yield the following two results ($0 \le t \le 1/2$):

$D'_2 = A_0 + tD_2 - tD_3$
$D'_3 = A_0 - tD_2 + tD_3$

This way, we obtain new subdivision coefficients (the corresponding subdivision matrix is in the Appendix).

It is also important to note that these subdivision coefficients need to be modified only in the immediate vicinity of the irregular vertex of the X-Shape, when calculating the F-vertex of the corresponding faces, and nowhere else. It is also important to note that this will take place in places where the symmetry condition holds, and nowhere else.

5.3. Subdivision of an X-Complex

Now, the subdivision of the X-Complex proceeds as usual, except that, when it comes to calculating its F-vertex, each X-Slice of the X-Configuration is replaced by its associated virtual face.

More specifically, every step of the CC subdivision routine proceeds as usual except when it comes to calculating the F-vertex of an X-Slice F with a starting vertex A_0. Here, we proceed as follows:

Assume that F = [D_2, A_0, E_1, E_2] is the X-Slice at the *lhs* of the intended polygonal complex and F' = [E_3, E_1 A_0,D_3] is the X-Slice at the *rhs* of that. Replacing F and F' by their associated virtual faces, the F-vertex coming from F is

$$(2A_0+tD_2-tD_3+E_1+E_2)/4$$

and that coming from F' is

$$(2A_0-tD_2+tD_3+E_3+E_1)/4$$

The reader may note that the parameter t may be used a shape handle to control the quality of the interpolating surface around the center of the X-Configuration.

Now, it is quite straightforward to expect that the one-step subdivision of an X-Complex is an X-Complex (see Figure 7) and the limit of subdivision of an X-Complex is a set of curves that start from the limit point of its associated X-Configuration.

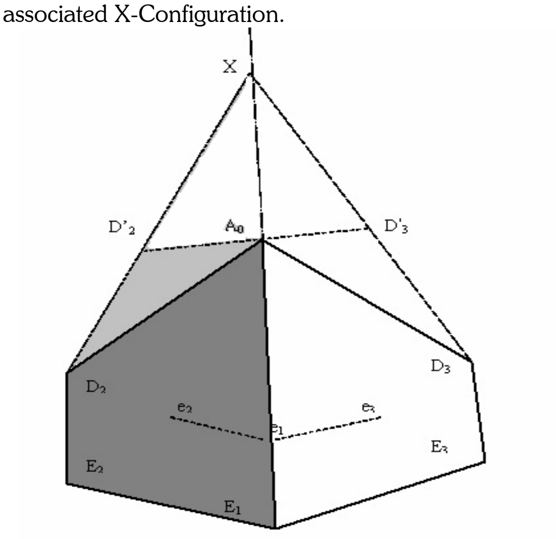


Fig. 9. The Construction

That is, the limit curves of an X-Complex is interpolated by any surface embodying this X-Complex and subdivided as indicated above (see Figure 12).

5.4. Continuity Analysis

The virtual face construction can be looked at in two ways. First, due to the nature of the X-Configuration, these faces turn out to be overlapping faces to which a normal Catmull-Clark mask is applied. As subdivision proceeds, this overlapping does not produce any undesirable side-effects because the centroid of the virtual face of an X-Slice will always fall within the X-Slice's boundaries. On the other hand, one may look at the virtual faces as modification of the subdivision coefficients around the point where the interpolated curves intersect, which means that continuity has to be analyzed. In this context, it is not difficult to conjecture that the smoothness condition will be maintained.

6. FURTHER WORK

The immediate work that needs to be done is an analysis of the subdivision matrix suggested above. In fact, the Eigen values will have to be computed and this analysis will have to be accompanied with pictures of the characteristic map.

As further work, we will project below how to construct a surface interpolating a set of intersecting curves.

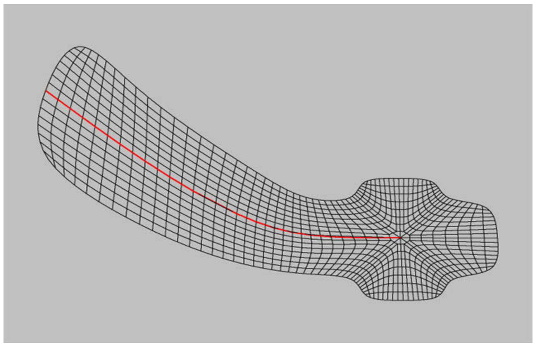


Fig. 10. The interpolated curve starts from the center of a planer X-Configuration

In fact, given a set of curves intersecting at a given point, it is enough to construct an X-Complex (X) that satisfies the following conditions:

- Each of the given curves is embodied in a polygonal complex (as its middle row) of (X).
- The X-Configuration of (X) satisfied the properties of planarity and symmetry stated above.

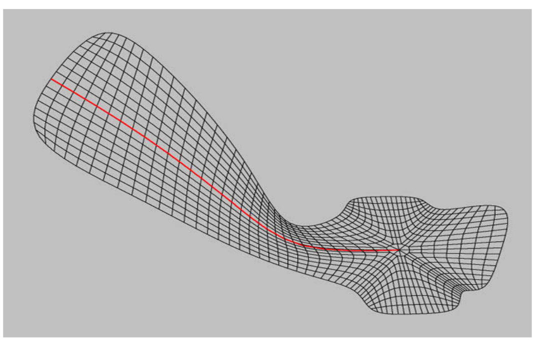
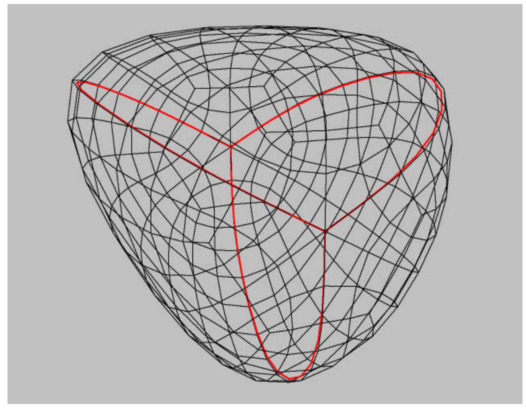


Fig. 11. The interpolated curve starts from the center of a non-planer X-Configuration

Having that, the repositioning of vertices as stated in equation(2) will be sufficient to guarantee that any surface embodying the resulting X-Complex will in fact be interpolating the initial curves.

Note here that the repositioning mentioned in previous paragraph does not at all change the properties of planarity and symmetry of the corresponding X-Configuration (see Figure 13).



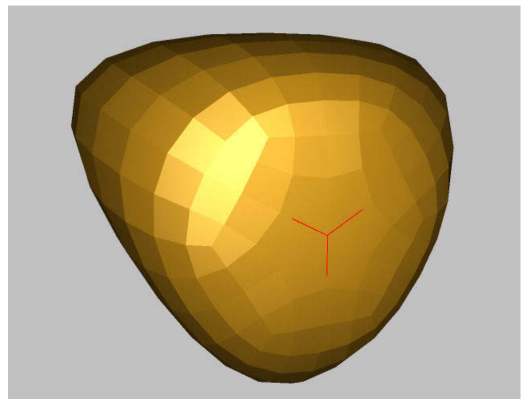


Fig. 12. Three interpolated curves meet at two different crossing points on a closed surface

7. CONCLUSIONS

A solution to the interpolation of more than two curves meeting at the same point in Catmull-Clark subdivision surfaces is presented. In addition, the approach can also handle dangling curves.

One advantage of using polygonal complexes here is in providing the ability to add derivative information across the interpolated curves. Therefore tangent plane and curvature constraints can be accommodated within the interpolation process. The symmetry conditions established around extraordinary points can be waved under affine map transformations leading to more freedom in the layout of the interpolated curves. Further work includes the construction of these curves from tagged control polygons on the polyhedron defining the surface and the use of the additional shape parameter around the intersection points.

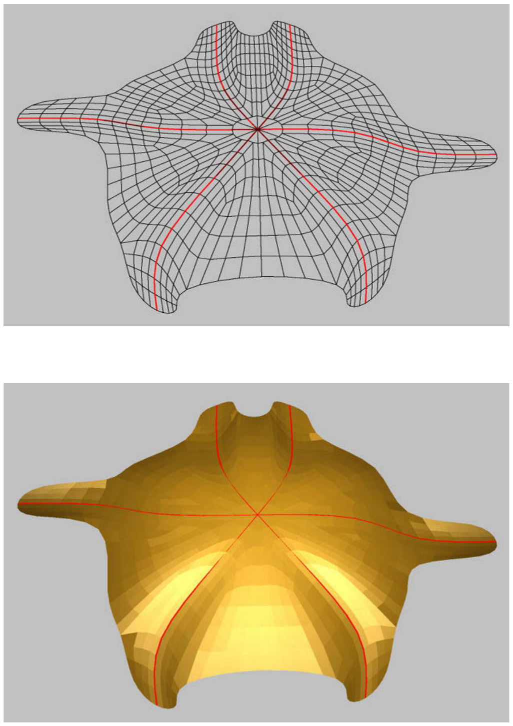


Fig. 13. Six different interpolated curves meet at the same crossing point on an open surface

8. REFERENCES

[1] Catmull, E. And Clark, J. Recursively Generated B-Spline Surfaces On Arbitrary meshes, Seminal Graphics, Ed. Rosalee Wolfe, ACM Press, Isbn 1-58113-052-X, pp. 183-188, 1998.

[2] Levin, A. Interpolating nets of curves by smooth subdivision surfaces, Proceedings of the 26th annual conference on Computer graphics and interactive techniques, pp.57-64, 1999.

[3] Nasri, A. Interpolation of Meshes of Curves by Recursive Subdivision Surfaces, presented at the Fourth SIAM conference on Geometric Design, Nashville, Nov. 6-9, 1995.

[4] Nasri, A. Curve Interpolation In Recursively Generated B-spline Surfaces Over Arbitrary Topology, Computer Aided Geometric Design Journal, Vol. 14, No 1, 1997.

[5] Nasri A., Recursive Subdivision of Polygonal Complexes and their applications in CAGD, Computer Aided Geometric Design Journal, Vol 17, pp 595-615, 2000.

[6] Nasri, A. and Abbas, A. Designing Catmull-Clark Subdivision Surfaces With Curve Interpolation Constraints, The Journal of Computers & Graphics, Vol 26, No. 3, 2002.

[7] Nasri, A., Abbas, A. & I. Hasbini, Skinning Catmull-Clark Subdivision Surfaces With Incompatible Cross-Section Curves, Pacific Graphics 2003, Canmore, Canada, October, 2003.

[8] Nasri, A. Interpolating an Unlimited Number of Curves Meeting at Extraordinary Points on Subdivision Surfaces, Computer Graphics Forum Journal, 22 (1), 87-97, 2003.

[9] Schweitzer, J. Analysis And Applications of Subdivision Surfaces, Ph.D. thesis, University of Washington, 1996.

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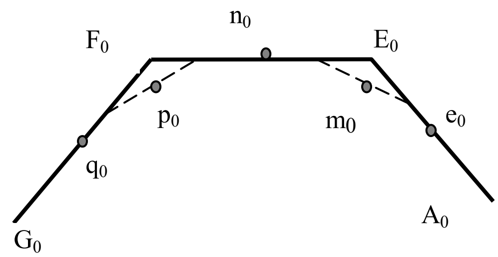

Fig. 14. CC Subdivision of a Polygon

APPENDIX

I. THE PROOF

This proof should be taken as a justification of the choice of the new subdivision coefficients around the centers of the X-Configurations of the mesh. The proof should be read with reference to Figures 7, 9 and 14.

In a CC subdivision step, a B-spline control polygon P = $[A_0,E_0,F_0,\ldots]$ is subdivided into a polygon Q = $[A_0,e_0,m_0,n_0,p_0,\ldots]$ as follows:

- e_0 (resp. n_0 and q_0) is the midpoint of the edge A_0E_0 (resp. E_0F_0 and F_0G_0), etc.
- m_0 (resp. p_0) is the midpoint of the edge connecting the midpoints of E_0e_0 and E_0n_0 (resp. the midpoint of the edge connecting the midpoints of F_0n_0 and F_0q_0), etc.
- The extremity A_0 of the open polygon is unchanged under this subdivision scheme

Repeating this subdivision step on the polygon Q (and so on) leads to polygons where consecutive vertices are closer and closer to each other. At the limit, this process converges to a smooth curve derived from the initial control polygon P.

We will only follow the behavior of the vertices that behave abnormally due to the irregularity of the centre vertex A_0.

a) The polygon:

- A_0 does not change
- $E_0 = (E_2 + 4E_1 + E_3)/6$
- $F_0 = (F_2 + 4F_1 + F_3)/6$
- $e_0 = (A_0 + E_0)/2$
  $= (A_0+E_2+4(A_0+E_1+A_0+E_3)/12$
- $m_0 = (e_0 + 2E_0 + n_0)/4$
  $= (A_0 + 6E_2 + F_2 + 4(A_0+6E_1+F_1)+ A_0+6E_3+F_3)/48$

b) The Subdivided Mesh

- $e_2 = (A_0 + E_1 + D_2 + E_2)/4$
- $e_3 = (A_0 + E_1 + D_3 + E_3)/4$
- $e_1 = (A_0 + E_1 + e_2 + e_3)/4$
  $= (D_2+E_2+6(A_0+E_1)+D_3+E_3)/16$
- $e_0 = (e_2+4e_1+e_3)/6$
  $= (D_2+E_2+4(A_0+E_1)+D_3+ E_3)/12$
- $m_2 = (A_0+D_2+6(E_1+E_2)+F_1+F_2)/16$
- $m_3 = (A_0+D_3+6(E_1+E_3)+F_1+ F_3)/16$
- $m_1 = (D_2+6E_2+F_2+6(A_0+6E_1+F_1)+D_3+ 6E_3 + F_3)/64$
- $m_0 = (m_2 + 4m_1 + m_3)/6$
  $= (D_2+6E_2+F_2+4(A_0+6E_1+F_1)+D_3+ 6E_3+F_3)/48$

A quick glance is enough to show that for the version of e_0 (resp. m_0) in the polygon to be equal to its counterpart in the subdivided mesh, it is sufficient to have:

$$A_0 = (D_2 + D_3)/2$$

II. THE SUBDIVISION MATRIX

The following points about this matrix should be noted:

- The matrix is multiplied by a factor of 1/288
- This matrix illustrates the case where the irregular vertex has a valence equal to 6. In Figure 15, we take u to be equal to 1 (also see section 5.2).

The matrix can be generalized through having a parameter t, where t = u/2 (see Figure 16). In other words: 0 <= t <= ½. (also see section 5.2)

\	A_0	E_1	B_3	B_2	D_2	D_3	A_1	E_2	E_3	C_3	A_3	A_2	C_2
a_0	216	12	12	12	8	8	8	2	2	2	2	2	2
e_1	144	108	0	0	0	0	0	18	18	0	0	0	0
b_3	144	0	108	0	0	0	0	0	0	18	18	0	0
b_2	144	0	0	108	0	0	0	0	0	0	0	18	18
d_2	144	18	0	18	90	-9	-9	18	0	0	0	0	18
d_3	144	18	18	0	-9	90	-9	0	18	18	0	0	0
a_1	144	0	18	18	-9	-9	90	0	0	0	18	18	0
e_2	144	72	0	0	36	-36	0	72	0	0	0	0	0
e_3	144	72	0	0	-36	36	0	0	72	0	0	0	0
c_3	144	0	72	0	0	36	-36	0	0	72	0	0	0
a_3	144	0	72	0	0	-36	36	0	0	0	72	0	0
a_2	144	0	0	72	-36	0	36	0	0	0	0	72	0
c_2	144	0	0	72	36	0	-36	0	0	0	0	0	72

Fig. 15. The Constant Subdivision Matrix (u = 1)

\	A_0	E_1	B_3	B_2	D_2	D_3	A_1	E_2	E_3	C_3	A_3	A_2	C_2
a_0	216	12	12	12	8	8	8	2	2	2	2	2	2
e_1	144	108	0	0	0	0	0	18	18	0	0	0	0
b_3	144	0	108	0	0	0	0	0	0	18	18	0	0
b_2	144	0	0	108	0	0	0	0	0	0	0	18	18
d_2	144	18	0	18	72 +36t	-18t	-18t	18	0	0	0	0	18
d_3	144	18	18	0	-18t	72 +36t	-18t	0	18	18	0	0	0
a_1	144	0	18	18	-18t	-18t	72 +36t	0	0	0	18	18	0
e_2	144	72	0	0	72t	-72t	0	72	0	0	0	0	0
e_3	144	72	0	0	-72t	72t	0	0	72	0	0	0	0
c_3	144	0	72	0	0	72t	-72t	0	0	72	0	0	0
a_3	144	0	72	0	0	-72t	72t	0	0	0	72	0	0
a_2	144	0	0	72	-72t	0	72t	0	0	0	0	72	0
c_2	144	0	0	72	72t	0	-72t	0	0	0	0	0	72

Fig. 16. The Parameterized Subdivision Matrix