# Optimal Single Biarc Fitting and its Applications

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**ABSTRACT**

This paper proposes a new approach for fitting an optimized biarc to a given 2D polygon and its two end tangents. A biarc can be constructed which matches two end points and two end tangents, but an additional constraint is required to make the biarc unique. The conventional approach to biarc construction, which has been adopted in arc spline approximation, introduces additional constraints to uniquely determine the biarc. Instead of imposing such constraints, the proposed approach exploits the inherent freedom in the choice of the biarc to achieve a better fit minimizing the distance between the polygon and the biarc. The approach is simple in concept and acceptable in computation. When applied in arc spline approximation tasks, the approach can play an important role in reducing the number of segments in the resulting arc spline. Some experimental results demonstrate its usefulness and quality.

**Keywords:** Biarcs; Single Biarc Fitting; Optimization; Arc Spline Approximation.

## 1. INTRODUCTION

Arc spline is a kind of geometric curve made of circular arcs and line segments. It is easy to use, computationally efficient in shape modeling, and well used as the description of tool path of CNC machines. In programming the CNC tool path, fewer arc segments can help to improve the production efficiency by reducing the number of instructions and tool motions [1-15]. Therefore, approximating points, polygons, or arbitrary curves with arc splines is of particular importance. Approximations to point data by $G^1$ arc splines have been investigated in the past [1-8]. Approximations to given curves by $G^1$ arc splines have also been extensively researched [9-15].

Biarcs have been widely used for such arc spline approximation. A biarc consists of two circular arcs with $G^1$ continuity at a joining point. Given two end points and two end tangents, a biarc can be constructed which matches the points and tangents, but an additional constraint is required to make the biarc unique. The conventional approach to biarc construction, which has been adopted in arc spline approximation, introduces various additional constraints to uniquely determine the biarc. The difference of the radii of the two circular arcs can be minimized, with the result that the angles of the two arcs are equal [1]. The difference of the curvatures of the two arcs can be minimized, with the result that the joining point is on the bisector of the line segment joining the two end points [1],[5],[7]. Also, the ratio of the two radii can be made as close to 1 as possible, with the result that the tangent at the joining point is parallel to the line joining the two end points [1],[2],[10].

This paper proposes a new approach for fitting an optimized biarc to a given 2D polygon and its two end tangents. Instead of imposing such constraints to make the biarc unique, the proposed approach exploits the inherent freedom in the choice of the biarc to achieve a better fit minimizing the distance between the polygon and the biarc. The approach is simple in concept, and it is computationally acceptable since the distance between a polygon and its fitted biarc can be computed directly and precisely. When applied in arc spline approximation tasks, the approach can reduce the number of segments in the arc spline much better than the conventional one. Some experimental results demonstrate its usefulness and quality.

The rest of the paper is organized as follows. In Section 2, a biarc formulation is given briefly. Section 3 describes the details on the optimal single biarc fitting. Section 4 describes the application of the proposed approach in two approximation tasks: $G^1$ arc spline approximation of 2D point data and $G^1$ arc spline approximation of a planar curve. Section 5 closes the paper.

## 2. BIARC FORMULATION

A biarc consists of two smoothly connected circular arcs that interpolate two end points and two end tangents [1],[5],[8],[14],[15]. Given two points $\mathbf{p}_s$ and $\mathbf{p}_e$, and two unit tangents $\mathbf{t}_s$ and $\mathbf{t}_e$, a biarc is sought so that it passes through $\mathbf{p}_s$ and $\mathbf{p}_e$, and it is tangential at $\mathbf{p}_s$ to $\mathbf{t}_s$ and at $\mathbf{p}_e$ to $\mathbf{t}_e$, and the arcs join in $G^1$ continuity. We adopt the biarc formulation presented in Refs. [8],[14]. As shown in Fig. 1, a biarc can be defined by five control points $\mathbf{p}_i^w = [w_i\mathbf{p}_i \quad w_i]^T \quad (i = 0,...,4)$.

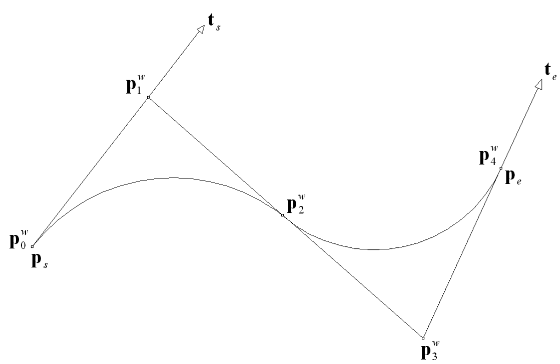



Fig. 1. Biarc formulation.

The unknown control points $\mathbf{p}_1^w$, $\mathbf{p}_2^w$, and $\mathbf{p}_3^w$ are sought. After the Euclidean projections $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are computed, weights are assigned to them. Since the end tangents are assumed to be of unit lengths, we get the following conditions:

$$\mathbf{p}_1 = \mathbf{p}_0 + \alpha\mathbf{t}_s, \qquad \mathbf{p}_3 = \mathbf{p}_4 - \beta\mathbf{t}_e,$$
$$\mathbf{p}_2 = \frac{\beta}{\alpha+\beta}\mathbf{p}_1 + \frac{\alpha}{\alpha+\beta}\mathbf{p}_3, \tag{1}$$

where $\alpha$ and $\beta$ are positive values. The three points $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ can be computed by solving the following equation:

$$\mathbf{v}\cdot\mathbf{v} + 2\mathbf{v}\cdot(\alpha\mathbf{t}_s + \beta\mathbf{t}_e) + 2\alpha\beta(\mathbf{t}_s\cdot\mathbf{t}_e - 1) = 0, \tag{2}$$

where $\mathbf{v} = \mathbf{p}_0 - \mathbf{p}_4$. The only unknowns in Eqn. (2) are $\alpha$ and $\beta$. Various conditions can be imposed on the ratio $r = \alpha/\beta$ to have a unique solution [1],[2],[5],[7],[10]. When the ratio $r$ is specified, Eqn. (2) leads to the following quadratic equation:

$$\mathbf{v}\cdot\mathbf{v} + 2\beta\mathbf{v}\cdot(r\mathbf{t}_s + \mathbf{t}_e) + 2r\beta^2(\mathbf{t}_s\cdot\mathbf{t}_e - 1) = 0, \tag{3}$$

Special cases requiring four arcs appear when $\mathbf{t}_s\cdot\mathbf{t}_e = 1$ or $\mathbf{v}\cdot(r\mathbf{t}_s + \mathbf{t}_e) = 0$. When Eqn. (3) has a positive root, the unknown points in Eqn. (1) are uniquely determined. The weights are assigned as follows [8],[14],[16]:

$$w_0 = w_2 = w_4 = 1, \quad w_1 = \cos(\alpha_1), \quad w_3 = \cos(\alpha_2),$$

where $\alpha_1$ and $\alpha_2$ denote the half sweep angles of the first and the second arc, respectively.

## 3. OPTIMAL SINGLE BIARC FITTING

Consider a polygon $\mathbf{P}$ to be defined by a sequence of points $\mathbf{p}_i \quad (i = 0,...,n)$ and two end tangents to be given as $\mathbf{t}_s$ at $\mathbf{p}_0$ and $\mathbf{t}_e$ at $\mathbf{p}_n$. The basic idea of the proposed approach is to fit a biarc $\mathbf{B}$ to the polygon $\mathbf{P}$ and its two end tangents in a way that minimizes the distance between the polygon and the biarc. Recall that different biarcs can be obtained with different ratios $r = \alpha/\beta$ in Eqn. (3). We can consider a mapping $\mathbf{B}(r)$ that creates a biarc $\mathbf{B}$ by solving Eqn. (3) with the ratio $r$. An optimized biarc can be obtained by finding an optimal ratio $\hat{r}$ such that $dist(\mathbf{P}, \mathbf{B}(\hat{r})) = \min_{r\in R} dist(\mathbf{P}, \mathbf{B}(r))$.

The optimal value $\hat{r}$ is probably found by nonlinear optimization searches [17]. Its approximation can be obtained by an iterative approach of progressively limiting the promising domain of the ratio. The approach divides the domain of interest at discrete ratio values, finds the ratio value leading to a biarc with the minimum distance, and replaces the domain by a smaller sub-domain in the vicinity of the ratio value. It repeats these steps until the range of the domain is smaller than a desired value. It is recommended to start with dense discrete values and decrease their number as iteration proceeds. Described below is an iterative procedure for such optimal single biarc fitting.

**Input**: a polygon $\mathbf{P} = \{\mathbf{p}_0,...,\mathbf{p}_n\}$, two unit tangents $\mathbf{t}_s$ and $\mathbf{t}_e$.

**Output**: a flag *f* for indicating whether a biarc can be fitted and, if possible, the fitted biarc $\mathbf{B}$.

(Step 1) If the two end points $\mathbf{p}_0$, $\mathbf{p}_n$ and the two tangents $\mathbf{t}_s$, $\mathbf{t}_e$ lead to one of special cases requiring four arcs, quit the procedure with the flag *f* set to *false*.

(Step 2) Initialize an integer value *d* and choose a real value $\rho$ where $0 < \rho < 1$. For instance, the values *d* and $\rho$ have been initialized as 10 and 0.25, respectively.

(Step 3) Initialize an upper limit $r_u$ of the ratio $\alpha/\beta$ in Eqn. (3). For instance, the value $r_u$ has been initialized as 5. Set its center $r_c$ and lower limit $r_l$ as 1 and $1/r_u$, respectively.

(Step 4) Compute the range $\Delta R$ of the current domain $R = R_l \cup R_r$ where $R_l = \{r \mid r_l \le r \le r_c\}$ and $R_r = \{r \mid r_c \le r \le r_u\}$, and choose $(2d+1)$ discrete ratio values $r_i$ $(0 \le i \le 2d)$ on the current domain.

(Step 5) For each ratio $r_i$, obtain a biarc $\mathbf{B}(r_i)$ interpolating the two end points $\mathbf{p}_0$, $\mathbf{p}_n$ and the two unit tangents $\mathbf{t}_s$, $\mathbf{t}_e$ by solving Eqn. (3) and compute the distance $dist(\mathbf{P}, \mathbf{B}(r_i))$ between the biarc $\mathbf{B}(r_i)$ and the polygon $\mathbf{P}$.

(Step 6) Find an index *j* leading to a biarc $\mathbf{B}(r_j)$ with the minimum distance.

(Step 7) If there is no such an index *j*, quit the procedure with the flag *f* set to *false*. If the current range is small enough compared to the initial range, quit the procedure with the biarc $\mathbf{B}(r_j)$ and the flag *f* set to *true*. For example, the ratio can be set to $10^{-4}$.

(Step 8) Replace the domain by a smaller one in the vicinity of the value $r_j$. If $j=0$, $r_l \leftarrow r_0$, $r_c \leftarrow r_1$, and $r_u \leftarrow r_2$. If $j = 2d$, $r_l \leftarrow r_{2d-2}$, $r_c \leftarrow r_{2d-1}$, and $r_u \leftarrow r_{2d}$. Otherwise, $r_l \leftarrow r_{j-1}$, $r_c \leftarrow r_j$, and $r_u \leftarrow r_{j+1}$.

(Step 9) Update the value *d* as $d = \max(2, \mathrm{int}((1-\rho)d))$ and go to Step 4.

In Step 4, the range $\Delta R$ is computed as $\Delta R = \Delta R_l + \Delta R_r$ where

$$\Delta R_l = \begin{cases} \dfrac{1}{r_l} - \dfrac{1}{r_c}, & 0 < r_l < r_c \le 1 \\ r_c - r_l, & 1 \le r_l < r_c \end{cases}$$

and

$$\Delta R_r = \begin{cases} \dfrac{1}{r_c} - \dfrac{1}{r_u}, & 0 < r_c < r_u \le 1 \\ r_u - r_c, & 1 \le r_c < r_u \end{cases}$$

Also, the ratio values $r_i$ are determined as follows:

$$r_i = \begin{cases} \left( \left(\dfrac{d-i}{d}\right)\dfrac{1}{r_l} + \left(\dfrac{i}{d}\right)\dfrac{1}{r_c} \right)^{-1}, & 0 < r_l < r_c \le 1 \\ \left(\dfrac{d-i}{d}\right) r_l + \left(\dfrac{i}{d}\right) r_c, & 1 \le r_l < r_c \end{cases}$$

and

$$r_{d+i} = \begin{cases} \left( \left(\dfrac{d-i}{d}\right)\dfrac{1}{r_c} + \left(\dfrac{i}{d}\right)\dfrac{1}{r_u} \right)^{-1}, & 0 < r_c < r_u \le 1 \\ \left(\dfrac{d-i}{d}\right) r_c + \left(\dfrac{i}{d}\right) r_u, & 1 \le r_c < r_u \end{cases}$$

for $i = 0,..,d$. Note that the values $\Delta R_l$, $\Delta R_r$, and $r_i$ are determined differently according as they are greater than 1 or not. For example, when $r_l = 1/5$, $r_c = 1$, $r_u = 5$, and $d = 4$, the ranges are determined as $\Delta R_l = \Delta R_r = 4$, and the ratio values are determined as {1/5, 1/4, 1/3, 1/2, 1, 2, 3, 4, 5}.

The procedure includes biarc fitting with the ratio $r = 1$. This implies its robustness since Eqn. (3) with $r = 1$ has a positive root in the general case and a valid biarc is created. Although the iterative approach taken herein requires more computation than the others interpolating two points and two tangents, this increase is acceptable since the direct computation of the distance between a biarc and a polygon is available.

### 3.1 Distance Metric in Single Biarc Fitting

The major benefit of using circular arcs is that the distance between a point and a circular arc can be computed precisely and, then, the distance between a polygon and its fitted biarc can be computed directly. Note that the polygon $\mathbf{P}$ and the biarc $\mathbf{B}$ have the same end points. Let the biarc $\mathbf{B}$ consist of two arcs $\mathbf{A}_l$ and $\mathbf{A}_r$. The distance between the polygon $\mathbf{P}$ and the biarc $\mathbf{B}$ can be defined in various ways. Two definitions of the distance are considered herein.

When the degree of mismatch between the biarc $\mathbf{B}$ and the polygonal points $\mathbf{p}_i$ is considered, the distance is defined as

$$dist(\mathbf{P}, \mathbf{B}) = \max_{i=0}^{n} dist(\mathbf{p}_i, \mathbf{A}_{j(i)}), \tag{4}$$

where the arc $\mathbf{A}_{j(i)}$ is the closest arc whose bounding rays contain the point $\mathbf{p}_i$. See Fig. 2. The distance between the point $\mathbf{p}_i$ and the arc $\mathbf{A}_{j(i)}$ is computed as

$$dist(\mathbf{p}_i, \mathbf{A}_{j(i)}) = \|\mathbf{p}_i - \tilde{\mathbf{p}}_i\| = rad_{j(i)} - \|\mathbf{p}_i - \mathbf{c}_{j(i)}\|, \tag{5}$$

where the point $\tilde{\mathbf{p}}_i$ is the orthogonal projection of the point $\mathbf{p}_i$ onto the arc, $rad_{j(i)}$ is the radius of the arc, and $\mathbf{c}_{j(i)}$ is its center.

When the degree of mismatch between the biarc $\mathbf{B}$ and all the points lying on the polygon $\mathbf{P}$ is considered, the distance is defined as

$$dist(\mathbf{P},\mathbf{B}) = \max_i dist(\mathbf{L}_i, \mathbf{A}_{k(i)}), \tag{6}$$

where the polygon $\mathbf{P}$ consists of a sequence of line segments $\mathbf{L}_i$ and the end points of each line segment $\mathbf{L}_i$ lie within the sweep angle of the same closest arc $\mathbf{A}_{k(i)}$ of the biarc $\mathbf{B}$. The distance between the line segment $\mathbf{L}_i$ and the arc $\mathbf{A}_{k(i)}$ is computed as

$$dist(\mathbf{L}_i, \mathbf{A}_{k(i)}) = \max\left(d_p, d_q, d_m\right), \tag{7}$$

where $d_p = \|\mathbf{p} - \tilde{\mathbf{p}}\|$, $d_q = \|\mathbf{q} - \tilde{\mathbf{q}}\|$,
and $d_m = \begin{cases} \|\mathbf{m} - \tilde{\mathbf{m}}\| & \text{if } \mathbf{m}_i \in \mathbf{L} \\ -1.0 & \text{otherwise} \end{cases}$.

See Fig. 2. The point $\mathbf{m}$ is the perpendicular projection of the center $\mathbf{c}$ onto the line segment $\mathbf{L}_i$. This result is consistent with that of Refs. [14],[15].

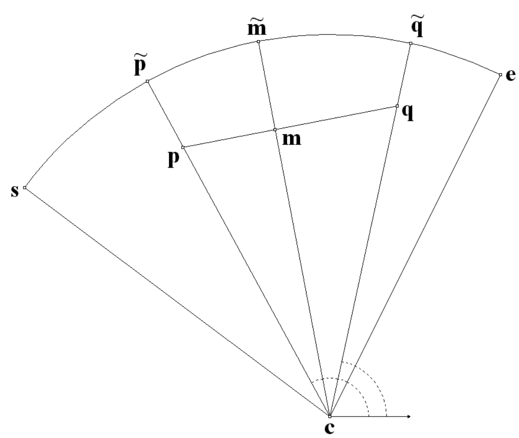


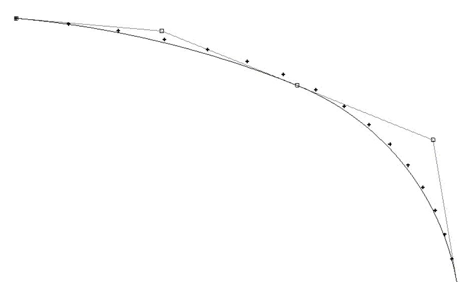

Fig. 2. Distance between a line segment and a circular arc.

The distance $dist(\mathbf{P},\mathbf{B})$ between the biarc $\mathbf{B}$ and all the points lying on the polygon $\mathbf{P}$ can be computed by first initializing the distance as zero and then repeating the following steps for each line segment $\mathbf{L}_i$:

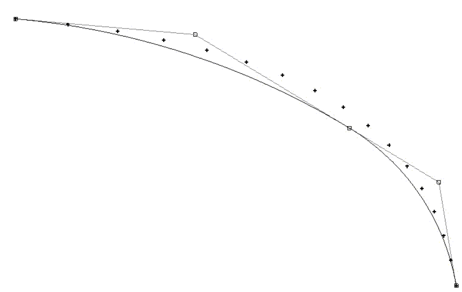
- If the two points of the line segment $\mathbf{L}_i$ lie within the sweep angle of the same closest arc $\mathbf{A}_{k(i)}$ for $k(i) \in \{l, r\}$, compute the distance $dist(\mathbf{L}_i, \mathbf{A}_{k(i)})$. Then, if $dist(\mathbf{L}_i, \mathbf{A}_{k(i)}) > dist(\mathbf{P},\mathbf{B})$, replace the distance $dist(\mathbf{P},\mathbf{B})$ by $dist(\mathbf{L}_i, \mathbf{A}_{k(i)})$.
- Otherwise, the line segment is divided with respect to the adjacent ray of the arcs and the new segments are tested in the same way.
- If both points are outside the sweep angles of all the arcs, the line segment is not accepted and $dist(\mathbf{L}_i, \mathbf{A}_{k(i)})$ is set to a large number.

### 3.2 Examples of Single Biarc Fitting

The proposed approach was tested with various sets of 2D polygonal data. Two examples are included to demonstrate its usefulness of quality. The distance between a polygon and its fitted biarc is computed using Eqn. (6). Fig. 3 shows the first example of single biarc fitting. The test polygon is enclosed in a $5.9 \times 3.6$ rectangle. Fig. 3(a) shows a biarc fitted with $r = 1$ where the distance between the polygon and the biarc is $7.228 \times 10^{-2}$. Fig. 3(b) shows a biarc obtained with Sabin's additional constraint that the tangent at the joining point should be parallel to the line joining the two end points [2],[10]. This corresponds to a biarc with $r = 1.726$, and the distance is $2.084 \times 10^{-1}$. Fig. 3(c) shows an optimal biarc obtained via the proposed approach which returns $\hat{r} = 0.784$ where the distance is $2.179 \times 10^{-2}$.



(a)



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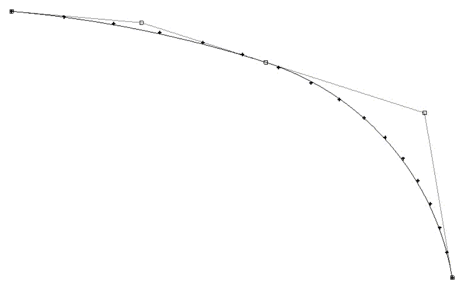
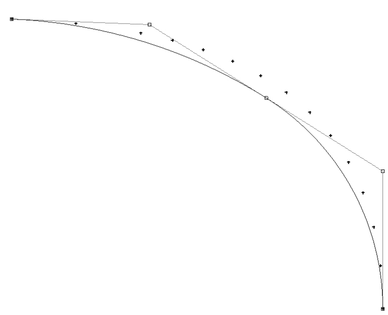


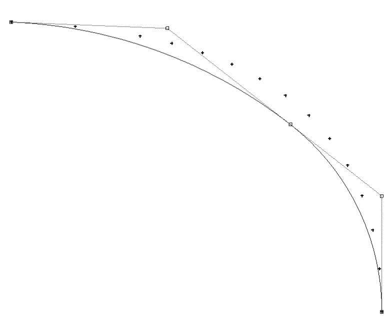
(c)

Fig. 3. First example: (a) biarc with $r = 1$; (b) biarc with Sabin's constraint; (c) biarc with $\hat{r} = 0.784$.

Fig. 4 shows the second example of single biarc fitting. The test polygon is enclosed in a $4.6 \times 3.6$ rectangle. Fig. 4(a) shows a biarc fitted with $r = 1$ where the distance is $2.004 \times 10^{-1}$. Fig. 4(b) shows a biarc obtained with Sabin's constraint. This corresponds to a biarc with $r = 0.741$, and the distance is $2.518 \times 10^{-1}$. Fig. 4(c) shows an optimal biarc fitted with $\hat{r} = 4.692$ where the distance is $7.024 \times 10^{-2}$.



(a)



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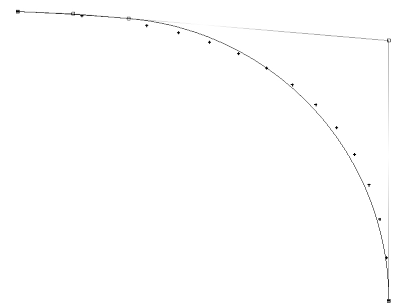


(c)

Fig. 4. Second example: (a) biarc with $r = 1$; (b) biarc with Sabin's constraint; (c) biarc with $\hat{r} = 4.692$.

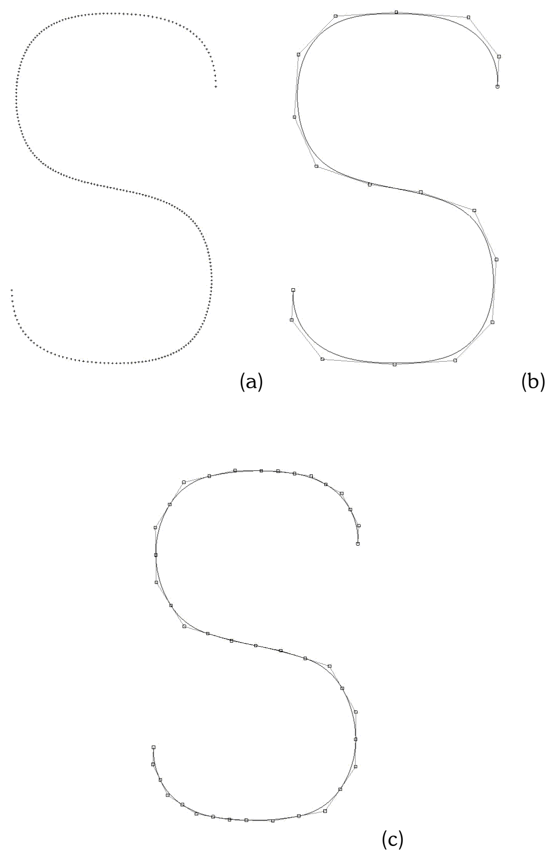
## 4. APPLICATIONS

The proposed approach for optimal single biarc fitting can be applied to several arc spline approximation tasks. Presented herein are its two practical applications: $G^1$ arc spline approximation of 2D point data and $G^1$ arc spline approximation of a planar curve. Their implementation has been done with C language on an IBM compatible personal computer with an Intel Pentium III processor.

### 4.1 Arc Spline Approximation of 2D Point Data

This task is to approximate 2D point data by a $G^1$ arc spline made of biarcs [1-8]. The point data includes a sequence of 2D points that form a polygon. Tangents at some points can be specified optionally. Tangents for approximation of discrete data can be estimated by some local methods or by B-spline curve fitting methods [8]. The idea used herein for estimating tangents is to approximate a cubic B-spline curve to the polygon within a tolerance [16]. We call it a base curve. The tangents at the points are obtained as follows: the points are perpendicularly projected onto the base curve; and the tangent directions are the derivatives of the base curve at them. A practical and simple approach for this $G^1$ arc spline approximation of 2D point data is to divide the point set of interest into smaller point subsets such that each point subset can be approximated by a biarc within a specified tolerance [5]. A point subset, which requires more than a biarc, can be simply divided into two at its median point. More details on $G^1$ arc spline approximation of 2D point data can be found in Refs. [5],[8].

This arc spline approximation was tested with various sets of 2D point data. Fig. 5 shows its application to a point set that forms an s-shaped polygon composed of 251 points. Its points are enclosed in a $10.0 \times 18.0$ rectangle. The tolerance is given as $10^{-1}$. The distance between a point subset and its fitted biarc is computed using Eqn. (4). Summarized in Tab. 1 are the results of arc spline approximation for six different tolerance levels. The results include the arc spline approximation using biarc fitting with the fixed ratio $r=1$ and with Sabin's constraint. Note that the approximation using the optimal biarc fitting requires the smaller number of circular arcs than the others. The optimal biarc fitting reduces the number of arcs by about 30% for biarc fitting with the fixed ratio, and about 35% for biarc fitting with Sabin's constraint.



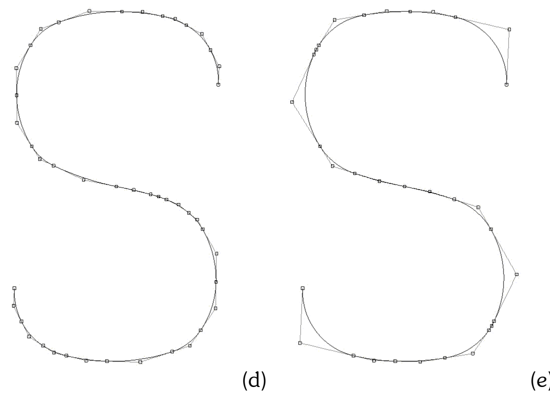




Fig. 5. Arc spline approximation to an s-shaped point set: (a) input point set; (b) cubic B-spline base curve with its control polygon; (c) biarcs obtained via biarc fitting with $r=1$; (d) biarcs obtained via biarc fitting with Sabin's constraint; (e) biarcs obtained via optimal biarc fitting.

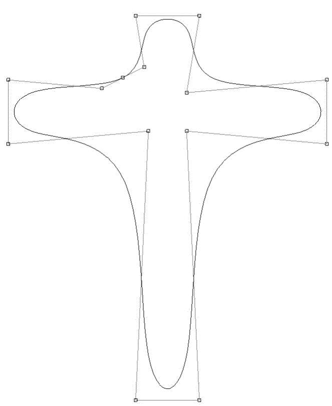
| Tol. | Numbers of circular arcs | | |
|---|---|---|---|
| | Biarc fitting with $r=1$ | Biarc fitting with Sabin's constraint | Optimal biarc fitting |
| $10^{-1}$ | 20 | 22 | 16 |
| $10^{-2}$ | 54 | 62 | 42 |
| $10^{-3}$ | 212 | 234 | 138 |
| $10^{-4}$ | 398 | 408 | 280 |
| $10^{-5}$ | 488 | 488 | 296 |
| $10^{-6}$ | 500 | 500 | 302 |

Tab. 1. Results of arc spline approximation of 2D point data.

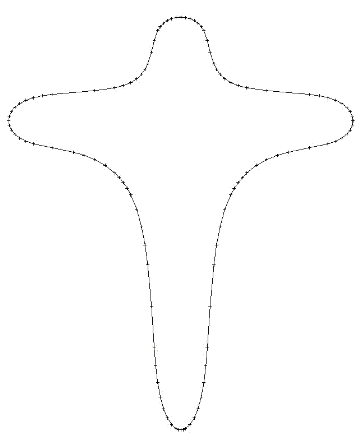
### 4.2 Arc Spline Approximation of a Planar Curve

This task is to approximate a planar parametric curve with a $G^1$ arc spline made of biarcs [9-15]. The practical approach for $G^1$ arc spline approximation of a planar curve is to divide the curve of interest into smaller segments such that each segment can be approximated by a biarc within a specified tolerance [14],[15]. A curve segment, which requires more than a biarc, can be simply divided into two at its median parameter. The biarc is obtained by polygonal approximation of the curve segment and single biarc fitting to the approximated polygon. The specified tolerance $\delta$ should be divided into two $\delta_1$ and $\delta_2$ where $\delta_1 = a\delta$, $\delta_2 = b\delta$ and $a+b=1$. The tolerance $\delta_1$ is for polygonal approximation and the tolerance $\delta_2$ for single biarc fitting. More details on $G^1$ arc spline approximation of a planar curve can be found in Refs. [14],[15].

This arc spline approximation was tested with various sets of planar curves. Fig. 6 shows its application to a closed cubic B-spline curve, which is cross-shaped and defined with 15 control points on a uniform knot vector. Its control points are enclosed in a $25.0 \times 30.0$ rectangle. The tolerance given as $10^{-1}$ is divided into two with $a$=0.2 and $b$=0.8. The distance between a polygon and its fitted biarc is computed using Eqn. (6). Summarized in Tab. 2 are the results of the tests for six different accuracy levels and two different tolerance distributions. The results also include the arc spline approximation using biarc fitting with the fixed ratio $r$=1 and with Sabin's constraint. Note that the approximation using the optimal biarc fitting requires the smaller number of circular arcs than the others. The optimal biarc fitting reduces the number of circular arcs by about 20% for biarc fitting with the fixed ratio, and about 21% for biarc fitting with Sabin's constraint.



(a)



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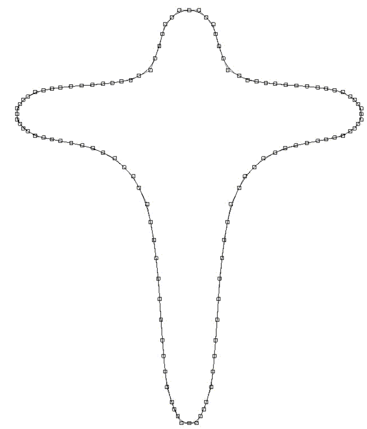

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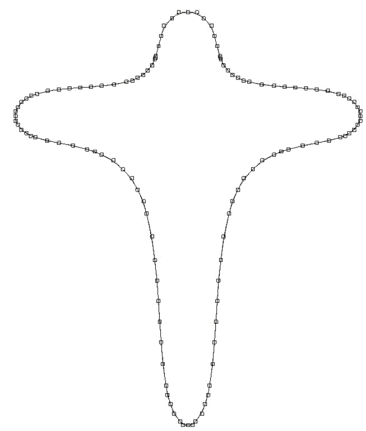

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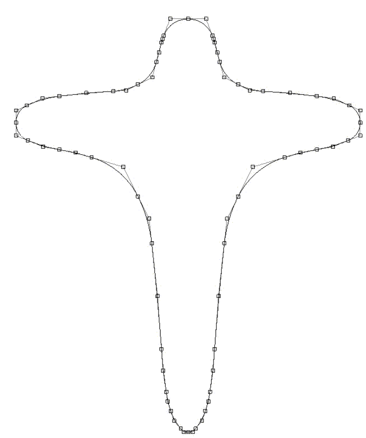


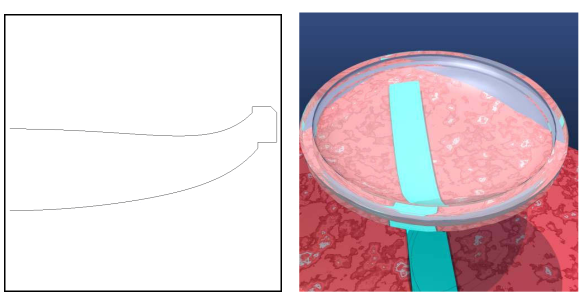
(e)

Fig. 6. Arc spline approximation applied to a cross-shaped curve: (a) cubic B-spline curve; (b) approximated polygon; (c) biarcs obtained via biarc fitting with $r = 1$; (d) biarcs obtained via biarc fitting with Sabin's constraint; (e) biarcs obtained via optimal biarc fitting.

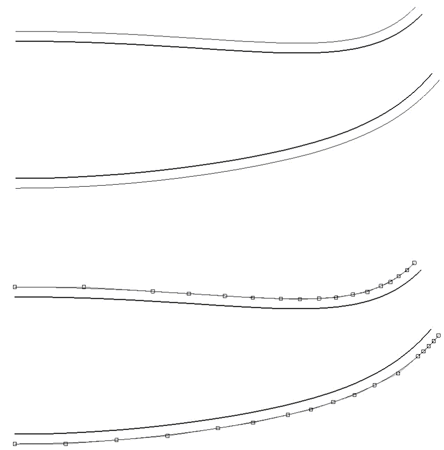
| Tol. distrib. (a,b) | Tol. | Poly. approx. (# of pts.) | Numbers of circular arcs | | |
|---|---|---|---|---|---|
| | | | Biarc fitting with $r$=1 | Biarc fitting with Sabin's constraint | Optimal biarc fitting |
| a=0.2 b=0.8 | $10^{-1}$ | 113 | 56 | 60 | 36 |
| | $10^{-2}$ | 371 | 96 | 108 | 76 |
| | $10^{-3}$ | 1073 | 196 | 196 | 164 |
| | $10^{-4}$ | 3603 | 392 | 400 | 352 |
| | $10^{-5}$ | 11837 | 856 | 864 | 796 |
| | $10^{-6}$ | 33917 | 1854 | 1854 | 1780 |
| a=0.5 b=0.5 | $10^{-1}$ | 69 | 68 | 64 | 48 |
| | $10^{-2}$ | 229 | 136 | 136 | 100 |
| | $10^{-3}$ | 749 | 304 | 300 | 216 |
| | $10^{-4}$ | 2137 | 684 | 672 | 484 |
| | $10^{-5}$ | 7219 | 1208 | 1208 | 1016 |
| | $10^{-6}$ | 23655 | 2632 | 2632 | 2096 |

Tab. 2. Results of arc spline approximation of a B-spline curve.

Arc spline approximation using optimal single biarc fitting can be usefully applied in CNC tool path generation for precision machining of aspheric lenses, which are used as important parts of various optical products [15]. For some aspheric lenses, their faces contain rotationally symmetric surfaces defined by revolving sectional curves a full $360°$ about an axis. Single point diamond turning operations are mostly performed to machine the molds that can later be used for producing the lenses via polymer injection molding [18]. As a very small tolerance is specified, fewer arcs are helpful to describe the tool path of CNC machines. Fig. 7 shows arc spline approximation applied to represent the offset curves of the sectional curves with $G^1$ arc splines made of biarcs. The sectional curves are enclosed in a $25.0 \times 25.0$ rectangle. The tolerance given as $10^{-3}$ is divided into two with $a$=0.2 and $b$=0.8. The distance between a polygon and its fitted biarc is computed using Eqn. (6).



(a) (b)



(c)

(d)

Fig. 7. Arc spline approximation applied in CNC tool path generation for precision machining of an aspheric lens: (a) 2D cross section of the lens; (b) rendered image of the lens; (c) offset curves of two major sectional curves; (d) biarcs obtained via optimal biarc fitting.

## 5. CONCLUDING REMARKS

This paper has presented a new approach for fitting an optimized biarc to a given 2D polygon and its two end tangents. While the conventional approach to biarc construction introduces additional constraints to uniquely determine the biarc, the proposed approach exploits the inherent freedom in the choice of the biarc to achieve a better fit minimizing the distance between the polygon and the biarc. The proposed approach is simple in concept and acceptable in computation. When applied in arc spline approximation tasks, the approach can play an important role in reducing the number of segments in the resulting arc spline. Some experimental results have demonstrated its usefulness and quality.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